Statistical Data Depth and its Applications

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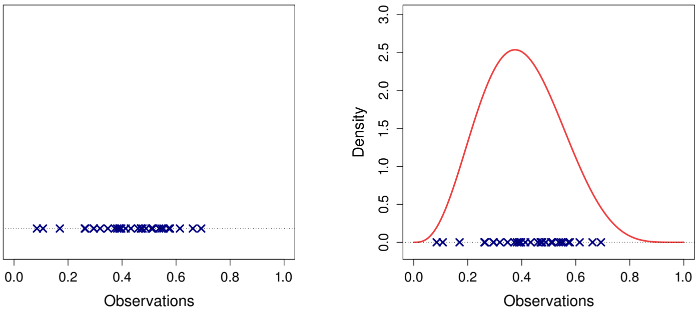
Introduction to Statistical Data Depth

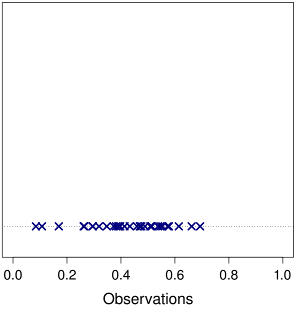
Univariate Statistical Model

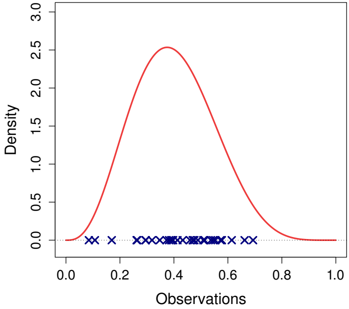
A random sample $X_1, \ldots, X_n$ of **univariate** observations ($\mathbf{\times}$)



Univariate Statistical Model

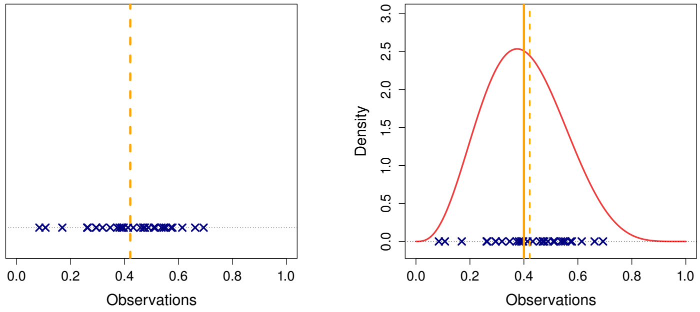
$X_1, \ldots, X_n \sim P \in \mathcal{P}(\mathbb{R})$ with a **density**





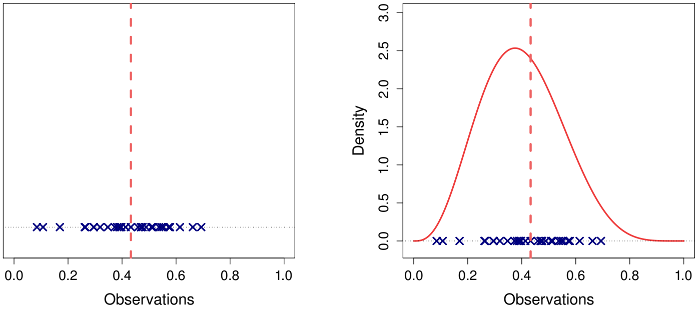
Location Estimation: Mean

Mean $\mathrm{E}\, X_1 = \int_{\mathbb{R}} x \,\mathrm{d}P(x)$ estimated by $1/n \sum_{i=1}^{n} X_i$



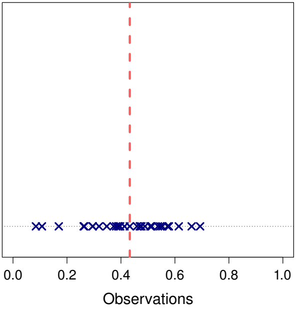
Location Estimation: Median

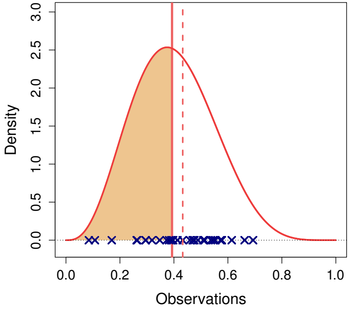
Sample median: the middle-most observation $X_{(n/2)}$



Quantiles for Univariate Data

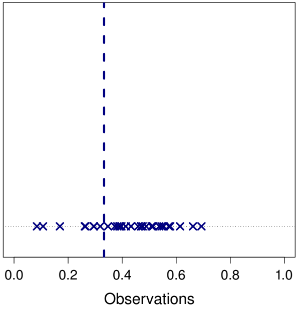
$$q(0.5) = \sup\{x \in \mathbb{R} \colon P((-\infty, x]) \leq 0.5\}$$

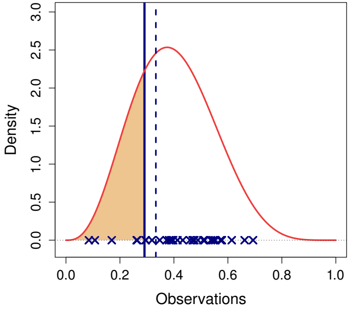




Quantiles for Univariate Data

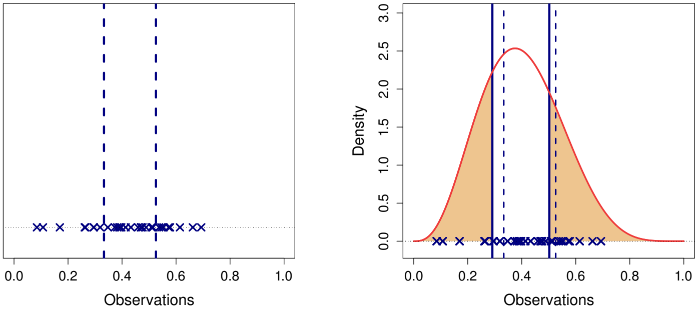
$$q(0.25) = \sup\{x \in \mathbb{R} \colon P((-\infty, x]) \leq 0.25\}$$





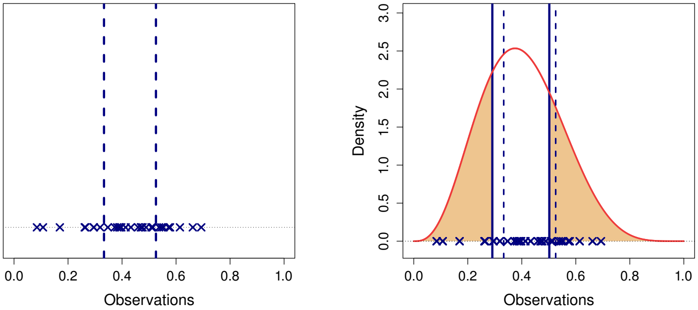
Quantiles for Univariate Data

$$q(0.75) = \sup\{x \in \mathbb{R} \colon P((-\infty, x]) \leq 0.75\}$$



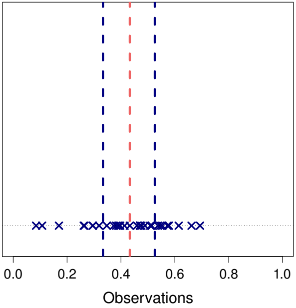
Quantiles for Univariate Data

$$IQR = q(0.75) - q(0.25)$$



Boxplot

Quantile-based **visualisation tool** (Tukey, 1969)



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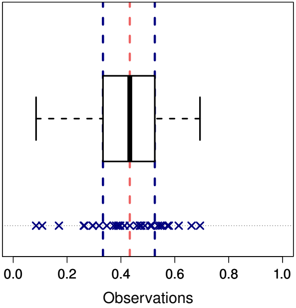
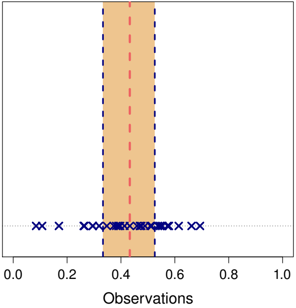


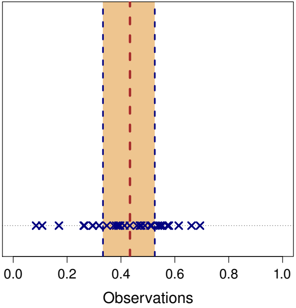
L-estimators

Central part of the data



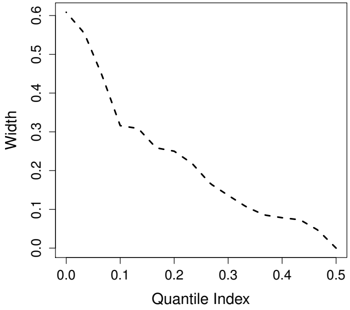
L-estimators

L-statistics: Functions of **order statistics** (trimmed mean)



Scale Curve

$$s\colon [0,1/2] \to [0,\infty) \colon t \mapsto q(1-t) - q(t)$$



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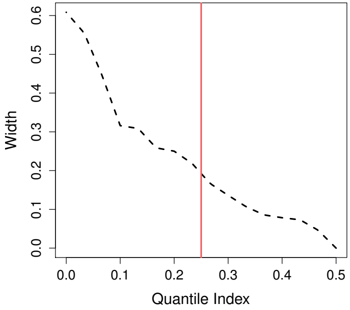
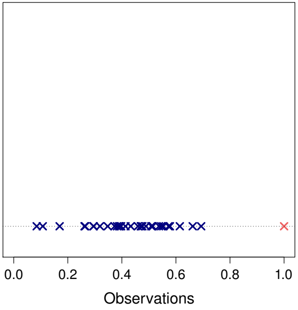


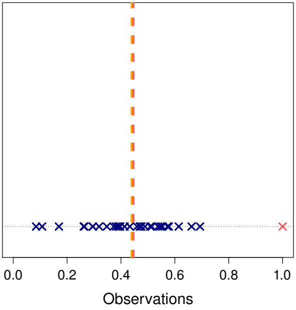
Outlier

Contaminate the dataset with an error $X_{n+1} = 1$



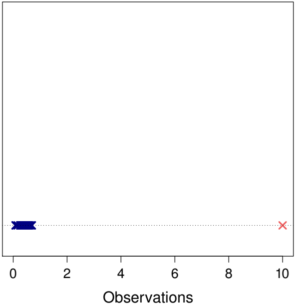
Outlier

Mean and median of the contaminated data



Severe Outlier

Contaminate with $X_{n+1} = 10$



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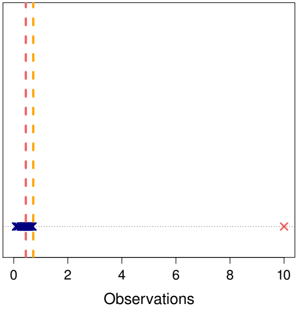
Mean and **median** of the contaminated data



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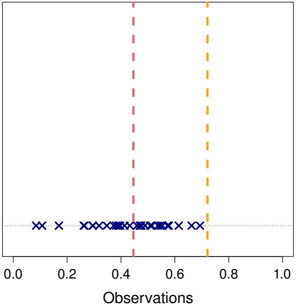
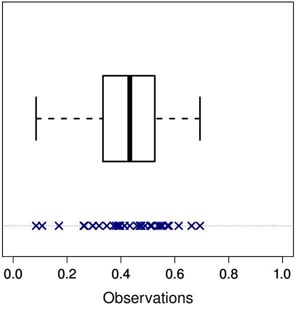


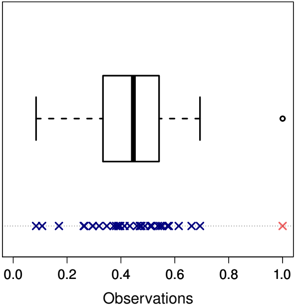
Boxplots

Boxplot of the **original data**



Boxplots

Boxplot of the **contaminated data**



Rank Tests: Two Sample Problem

Let $X_1, \ldots, X_n \sim P$ and $Y_1, \ldots, Y_m \sim Q$ be independent univariate random samples (no ties). Test

$$H_0 : P = Q \quad \text{against} \quad H_1 : P \neq Q.$$

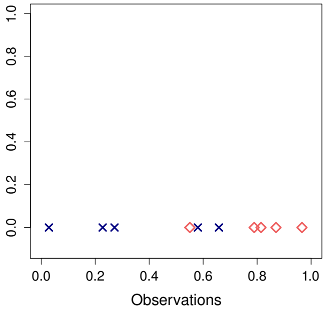
Wilcoxon's rank sum test (Wilcoxon, 1945):

- Pool the two samples into $Z_1, \ldots, Z_{n+m}$ and **rank** these observations (1 through $n + m$).
- Add up the ranks of those observations which came from the sample from P. Denote by R.
- Reject H_0 if R is either too small, or too large.

Wilcoxon's Rank Sum Test: Illustration

$$X \sim B(1,2), Y \sim B(2,1), n = m = 5$$

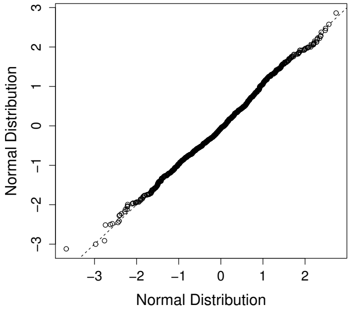
$R = 17$ (range from 15 to 40), p-value 0.03



Q-Q Plot

Quantile-versus-quantile plot (Gnanadesikan and Wilk, 1968)

$$t \mapsto (q_X(t), q_Y(t))$$



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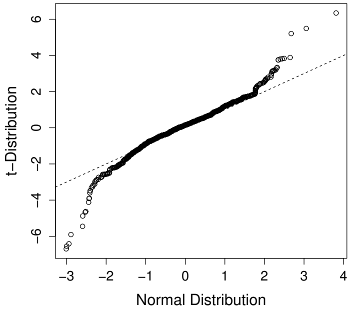
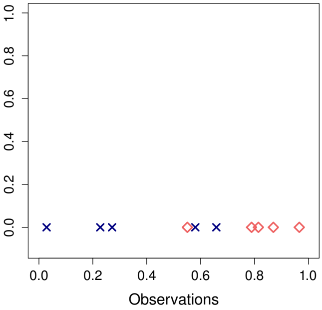
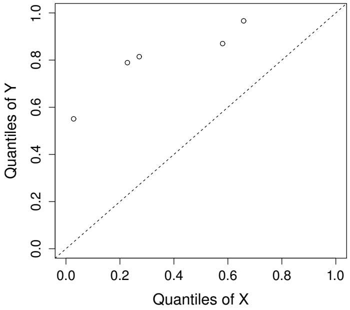


Wilcoxon's Rank Sum Test: Illustration

$$X \sim B(1,2), Y \sim B(2,1), n = m = 5$$

$R = 17$ (range from 15 to 40), p-value 0.03

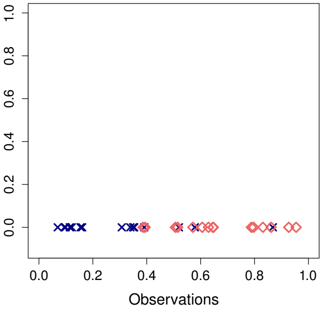


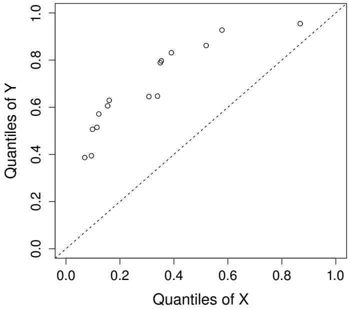


Wilcoxon's Rank Sum Test: Illustration

$$X \sim B(1,2), Y \sim B(2,1), n = m = 15$$

$R = 143$ (range from 120 to 345), p-value 0.00

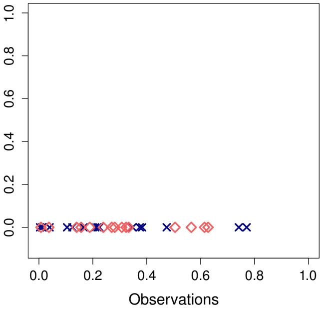


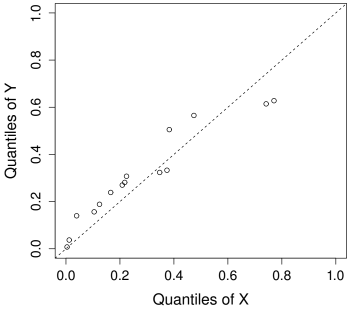


Wilcoxon's Rank Sum Test: Illustration

$$X \sim B(1,2), Y \sim B(1,2), n = m = 15$$

$R = 220$ (range from 120 to 345), p-value 0.62





Summary: Ranks and Orders

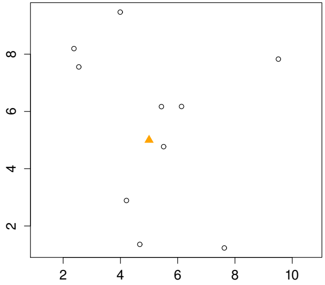
In $\mathbb{R}$, **rank** and **order statistics** enable:

- effective data **visualisation** (Q-Q plot);
- **outlier detection** (boxplot);
- construction of **robust** estimators (L-statistics);
- **non-parametric** data analysis (rank tests).

All thanks to the **linear ordering** on the sample space.

Multivariate Data

How to order **multivariate data**?



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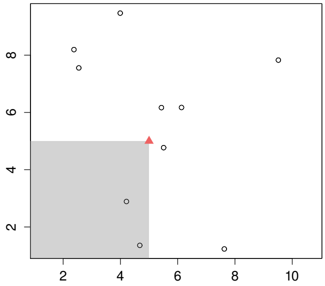

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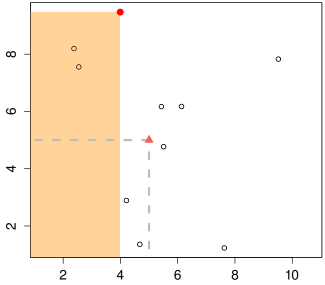

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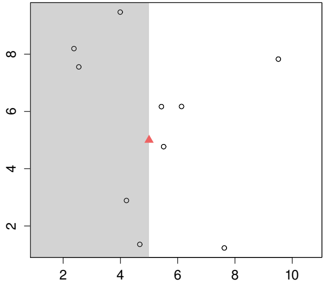

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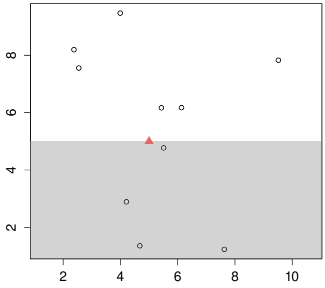

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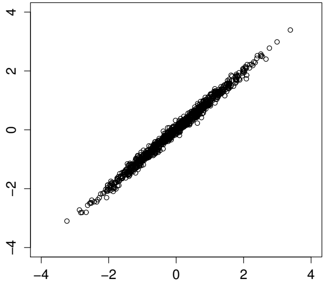

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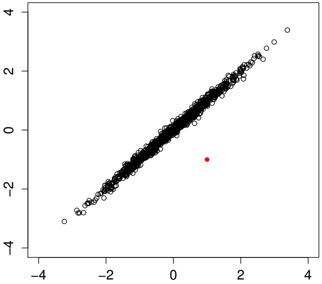

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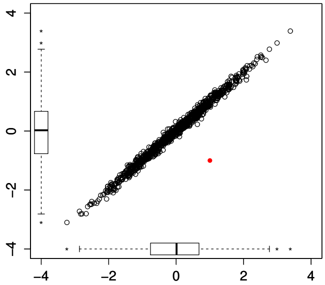

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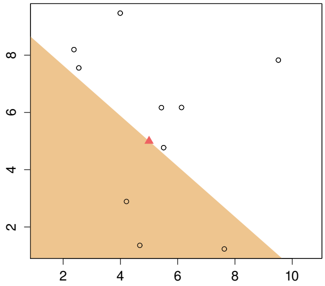
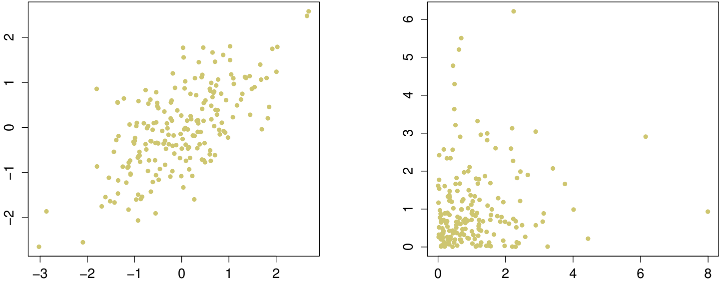


Data Depth

For a random variable $X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$, consider the **depth** of $x \in \mathbb{R}^d$ w.r.t. P

$$D\colon \mathbb{R}^d \times \mathcal{P}\left(\mathbb{R}^d\right) \to [0,1]\colon (x,P) \mapsto D(x,P).$$



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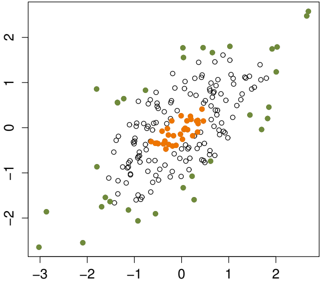
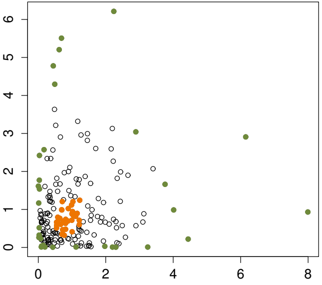
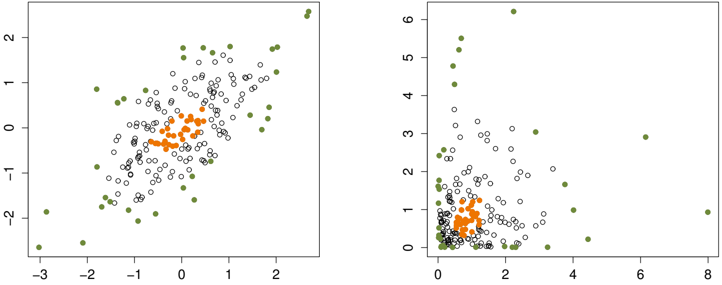




Halfspace Depth

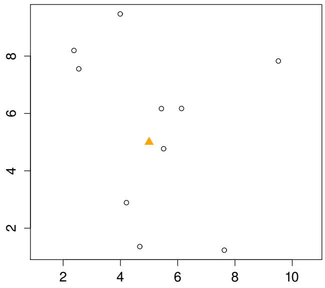
Halfspace depth (Tukey, 1975) of an observation in $\mathbb{R}^d$

$$hD(x;P) = \inf_{H \in \mathcal{H}(x)} P(H).$$



Halfspace Depth

$$hD(x; P_n) = \min \frac{\text{\# of observations in a halfspace that contains } x}{n}$$



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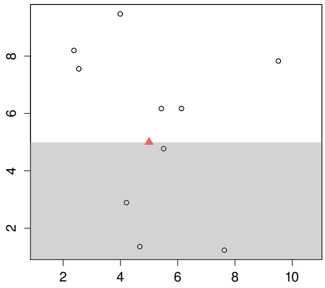

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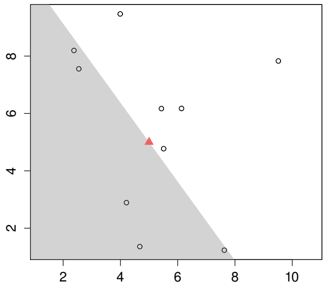

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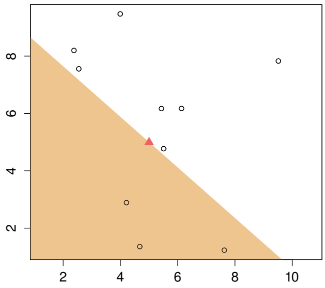


Brief History of *hD* (in Statistics)

1955 Idea with minimal halfspaces first used by Hodges;

1975 Tukey proposes *hD* as a visualisation tool;

1982 Donoho studies *hD* in his Ph.D. thesis;

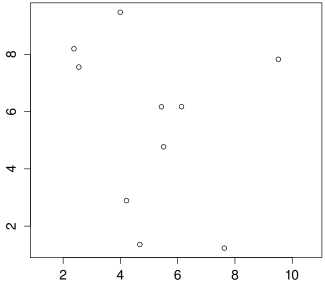
1992 depth introduced in AoS (Donoho and Gasko, 1992);

1999 Rousseeuw and Ruts study *hD* in full generality;

2000 Zuo and Serfling provide a general framework for the depth.

Depth Region

$$hD_\alpha(P) = \{x \in \mathbb{R}^d : hD(x;P) \geq \alpha\}$$



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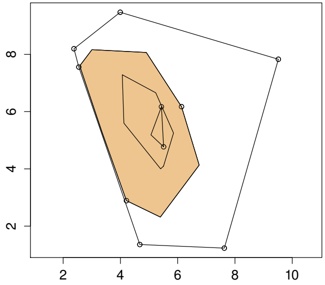
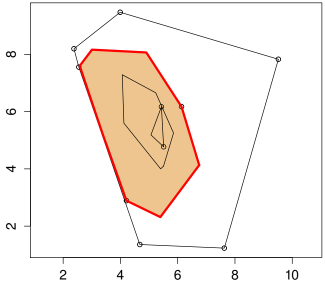


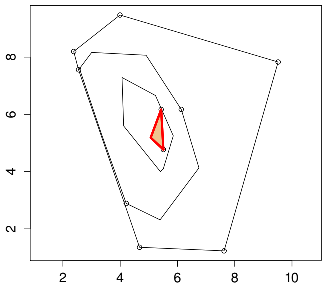
Depth Contour

Topological boundary of $hD_\alpha(P)$



Halfspace Median

Point(s) at which the depth $hD(\cdot; P)$ is maximized over $\mathbb{R}^d$



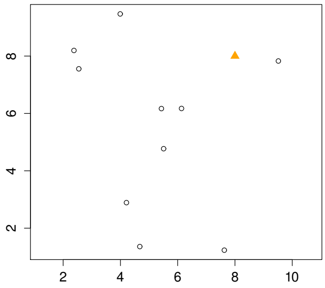
Elementary Properties

It holds true that

- $hD(x;P)$ is **well defined** for any $x \in \mathbb{R}^d$ and $P \in \mathcal{P}\left(\mathbb{R}^d\right)$;
- $hD(x;P) \in [0,1]$;
- a halfspace median **always exists**;

Minimizing Halfspace

The minimizing halfspace may **not be unique**



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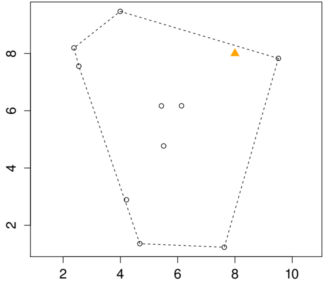

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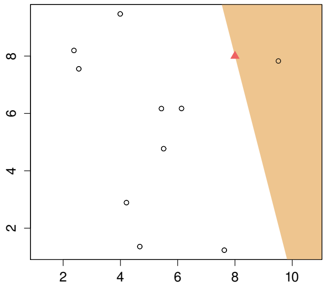

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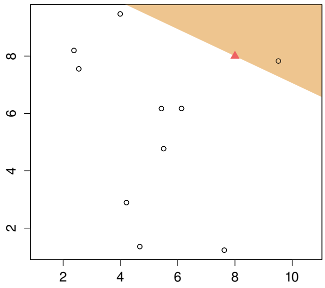
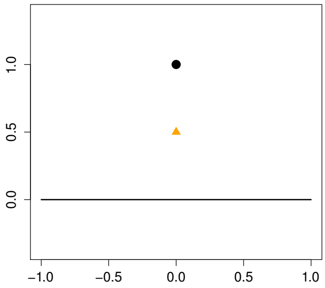


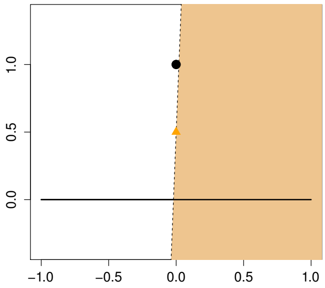
Minimizing Halfspace

The minimizing halfspace may **not exist**



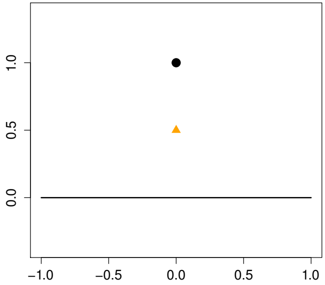
Minimizing Halfspace

The minimizing halfspace may **not exist**



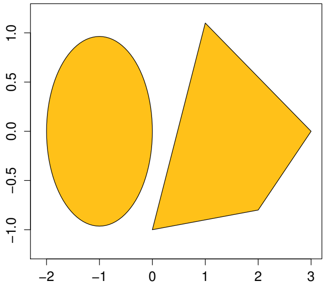
Assumption 1: Smoothness (S)

$P(\partial H) = 0$ for each halfspace H



Assumption 2: Contiguous Support (C)

The mass of P cannot be divided by a **slab of zero probability** (Mizera and Volauf, 2002)



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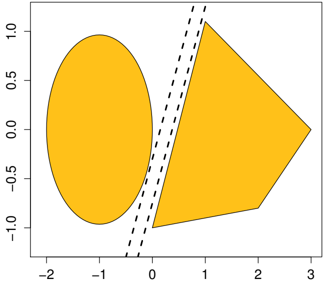

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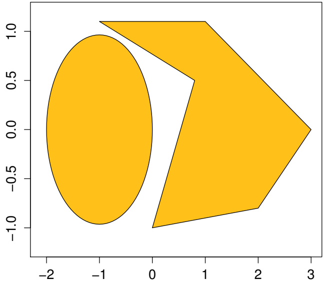


Further Properties

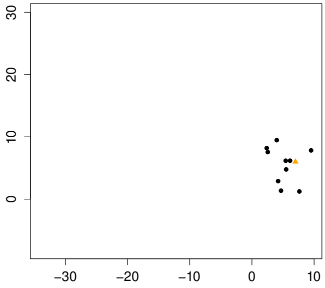
For P that **satisfies (S)**

- $hD(x; P) \in [0, 1/2]$;
- a minimizing halfspace **exists** at any $x \in \mathbb{R}^d$;
- if (C) is also true, the halfspace median is **unique**.

Affine Invariance

For any $A \in \mathbb{R}^{d \times d}$ non-singular and $b \in \mathbb{R}^d$

$$hD(x; P_X) = hD(Ax + b; P_{AX+b}).$$



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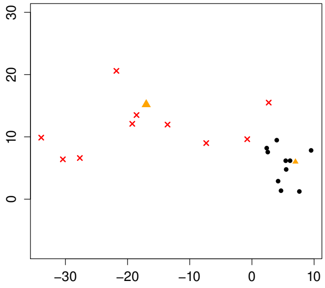

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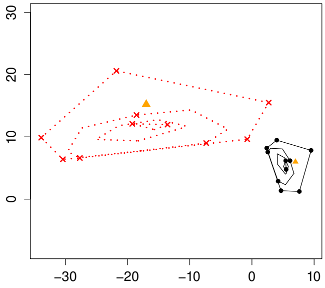
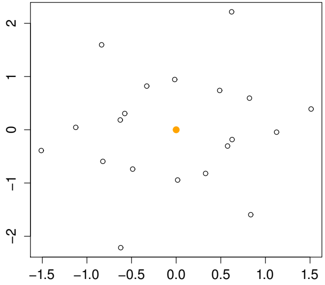


Maximality

If X is **symmetric** (i.e. $P_X = P_{-X}$), then

$$hD(0;P) = \sup_{x \in \mathbb{R}^d} hD(x;P).$$



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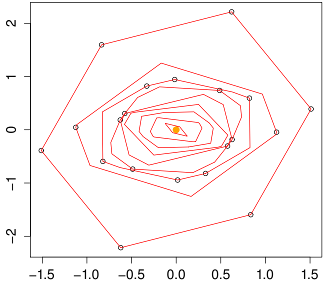


(Semi-)Continuity

Theorem (Mizera and Volauf, 2002)

For any $x_\nu \to x$ *in* $\mathbb{R}^d$ *and* $P_\nu \xrightarrow[\nu\to\infty]{w} P$ *in* $\mathcal{P}\left(\mathbb{R}^d\right)$

$$\limsup_{\nu\to\infty} hD(x_\nu;P_\nu) \leq hD(x;P).$$

In particular,

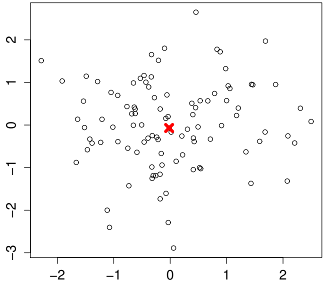
$$\limsup_{\nu\to\infty} hD(x_\nu;P) \leq hD(x;P).$$

If P *satisfies (S) then also*

$$\lim_{\nu\to\infty} hD(x_\nu;P_\nu) = hD(x;P).$$

Robustness

Halfspace median is a **robust estimator**



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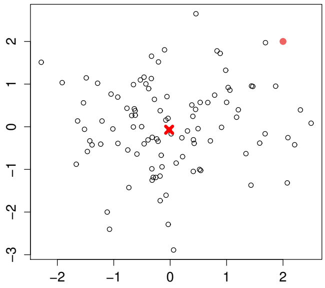

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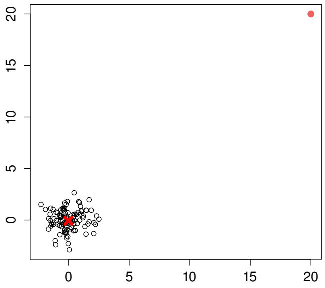

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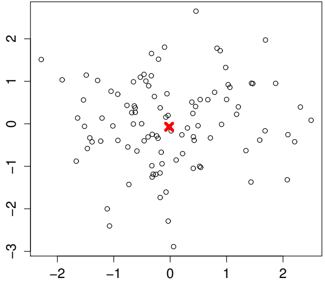


Sample Version Consistency

Theorem (Donoho and Gasko, 1992)

For any $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ *almost surely*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \left| hD(x;P_n) - hD(x;P) \right| = 0.$$

Vanishing at Infinity

Theorem (Donoho and Gasko, 1992)

For any $P \in \mathcal{P}\left(\mathbb{R}^d\right)$

$$\lim_{\|x\| \to \infty} hD(x; P) = 0.$$

Properties of Depth Regions

For each $\alpha > 0$ it holds true that (Rousseeuw and Ruts, 1999)

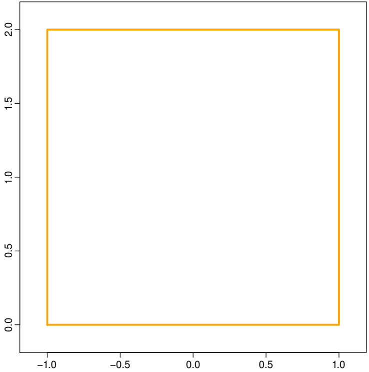
- $hD_\alpha(P) = \bigcap\{H \in \mathcal{H} : P(H) > 1 - \alpha\}$;
- $hD_\alpha(P)$ is **closed**;
- $hD_\alpha(P)$ is **bounded**;
- $hD_\alpha(P)$ is **convex**.

$hD(\cdot; P)$ is a **quasi-concave function** for any P.

Quasi-Concavity

hD is always **quasi-concave**, i.e. for each $\alpha \in [0,1]$

$$\left\{x \in \mathbb{R}^d : hD(x;P) \geq \alpha\right\} \text{ is a convex set}$$



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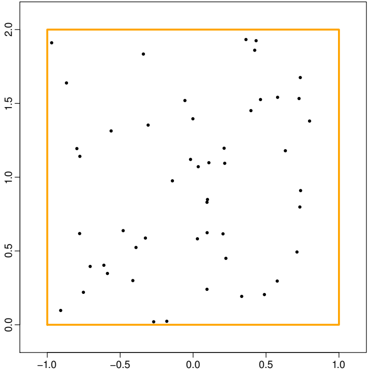

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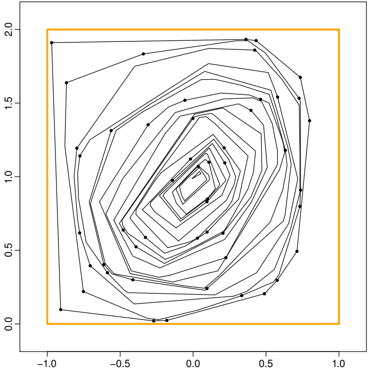

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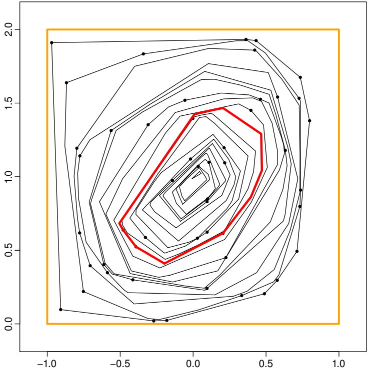
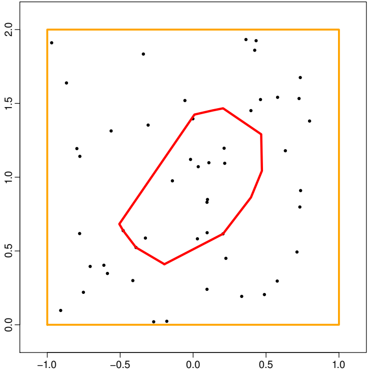


Consistency of Depth Regions

Consider the mapping

$$\alpha \mapsto \left\{x \in \mathbb{R}^d : hD(x;P) \geq \alpha\right\}$$



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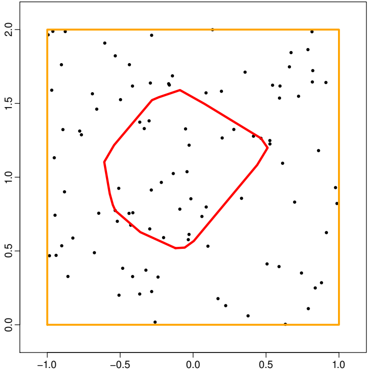

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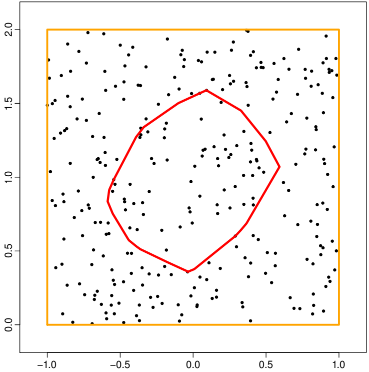

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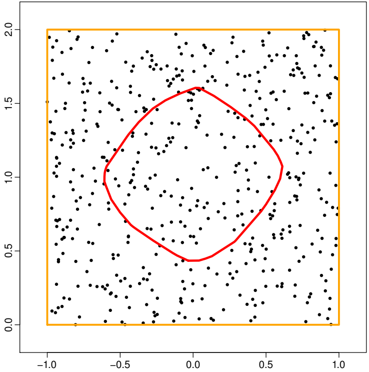

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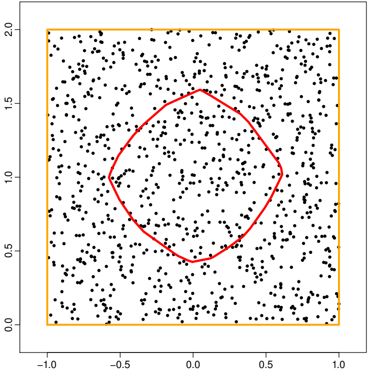

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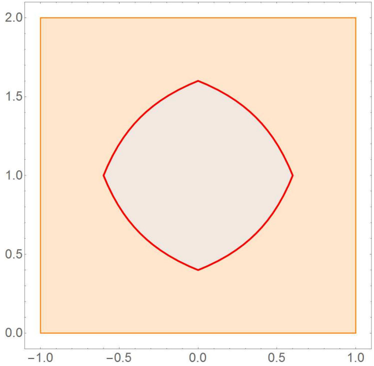


Properties of Depth Regions

Convex sets are equipped with the Hausdorff distance d_H.

Theorem (Dyckerhoff, 2017+)

Let (S) and (C) be true for P. Then the mapping

$$\alpha \mapsto hD_\alpha(P)$$

is continuous. Further, for any α

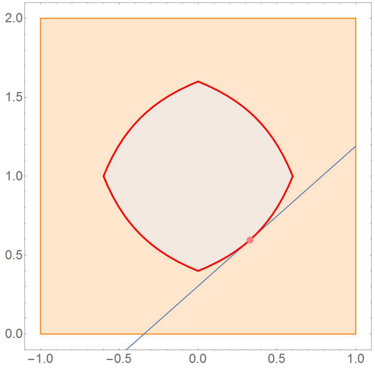
$$d_H\left(hD_\alpha(P_n), hD_\alpha(P)\right) \xrightarrow[n\to\infty]{\text{a.s.}} 0.$$

The previous results of Zuo and Serfling (2000b) are **not correct!**

Asymptotic Normality

$\sqrt{n}\, hD(x; P_n)$ is **asymptotically normal**

$\iff$ $hD(x; P)$ is realised by a **single halfspace** $H \in \mathcal{H}$ (Massé, 2004)



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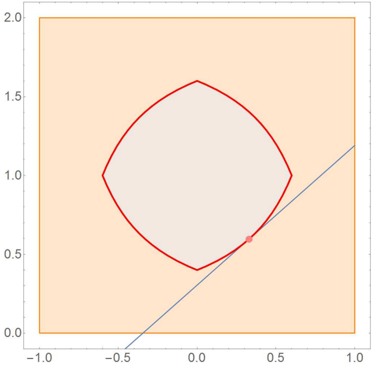
$\iff$ the contour of $hD(\cdot; P)$ is **smooth** at x (Gijbels and Nagy, 2016)



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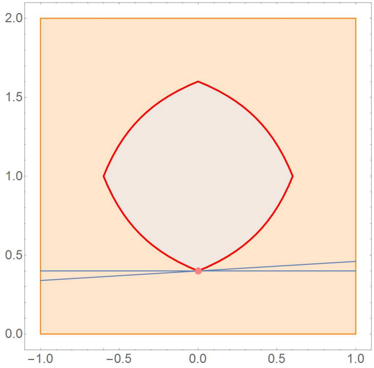
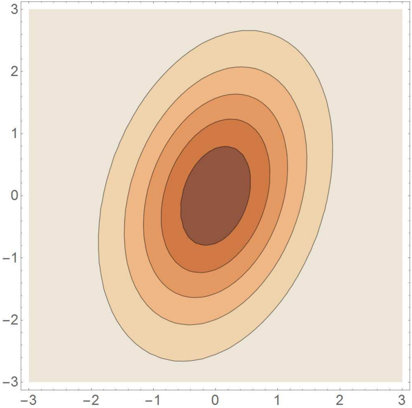


Population Depth: Elliptically Symmetric Distributions

Elliptically symmetric distributions have **elliptic depth contours**



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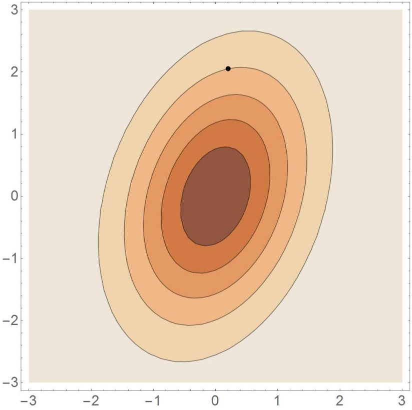

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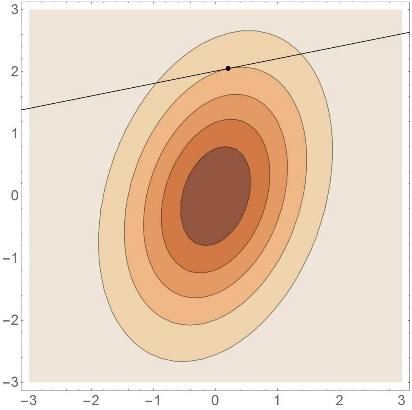
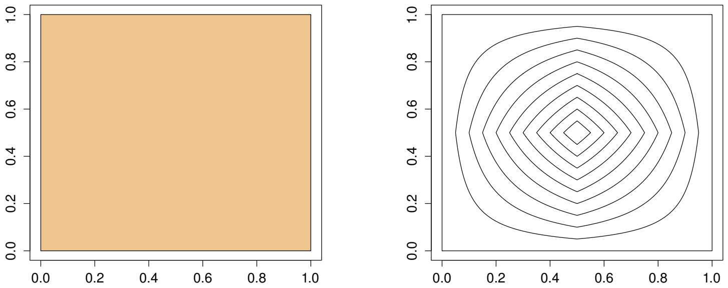


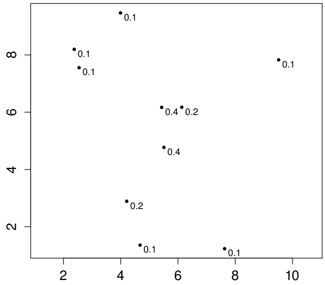
Population Depth: Uniform Distribution on a Square

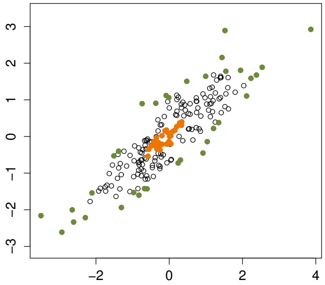
Uniform distribution on a simple convex body



Data Ordering

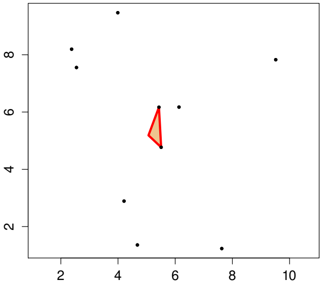
Depth induces a **centre - outward ordering** of points

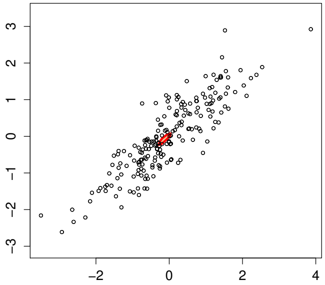




Halfspace Median

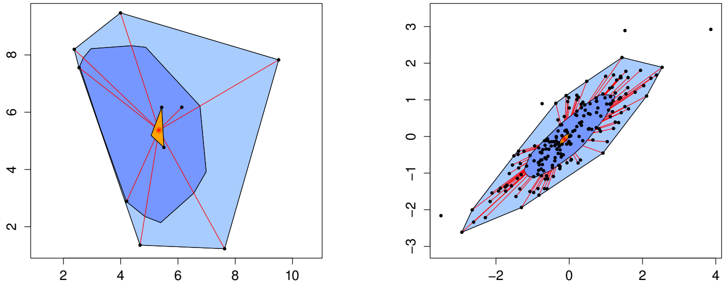
Point(s) that maximize the depth over $\mathbb{R}^d$





Bagplot: A Multivariate Boxplot

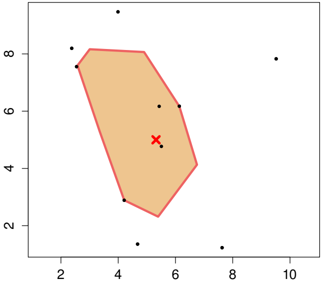
Central bag: 50% deepest observations (Rousseeuw et al., 1999)

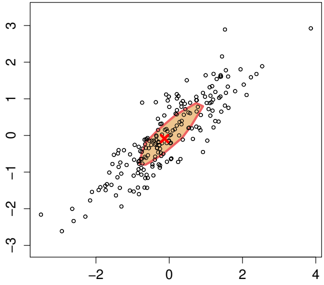


Multivariate L-statistics

Depth-trimmed mean (Fraiman and Meloche, 1999)

$$\sum_{i=1}^{n} X_i \, \mathbb{I}\left(hD(X_i; P_n) \geq \alpha\right) \Big/ \sum_{i=1}^{n} \mathbb{I}\left(hD(X_i; P_n) \geq \alpha\right)$$

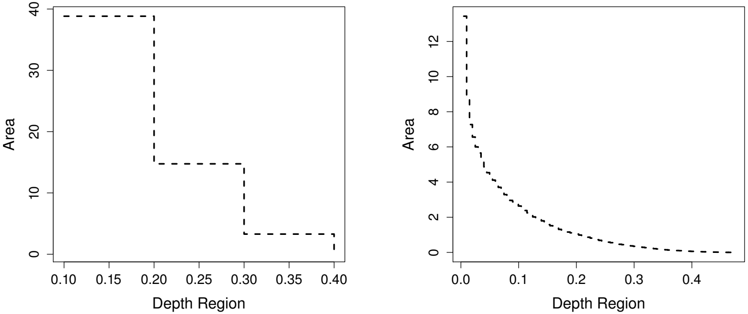




Scale Curve

Volume of the **depth region** (Liu et al., 1999)

$$s : [0,1] \to [0,\infty) : \alpha \mapsto \lambda(hD_\alpha(P))$$



Multivariate Rank Tests: Two Sample Problem

Let $X_1, \ldots, X_n \sim P$ and $Y_1, \ldots, Y_m \sim Q$ be independent **multivariate** random samples. Test

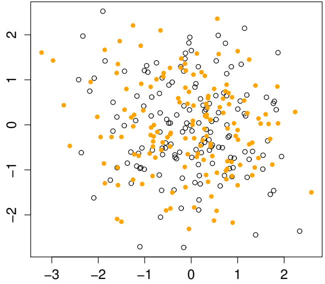
$$H_0 : P = Q \quad \text{against} \quad H_1 : P \neq Q.$$

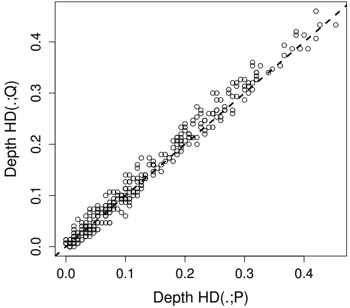
Wilcoxon's rank sum test (Liu and Singh, 1993):

- Pool the two samples into $Z_1, \ldots, Z_{n+m}$ and rank these observations by their **depth** (1 through $n+m$).
- Add up the ranks of those observations which came from the sample from P. Denote by R.
- Reject H_0 if R is either too small, or too large.

D-D Plots: Multivariate Q-Q Plots

Replace quantiles by **depth in Q-Q plots** (Liu et al., 1999)





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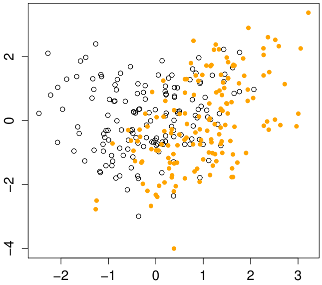
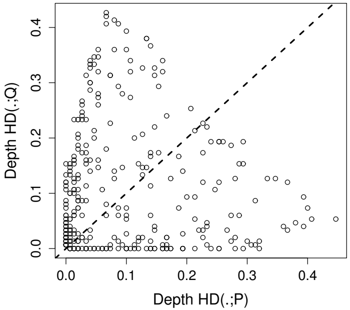
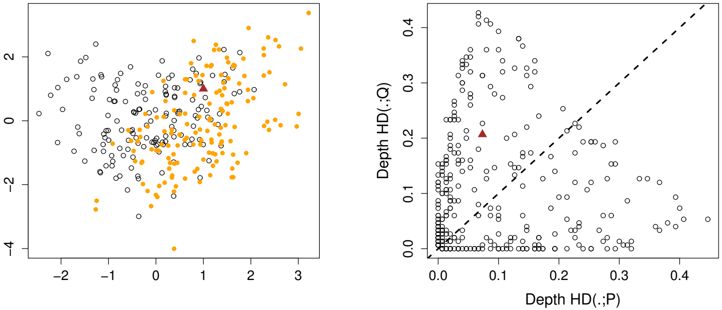




Classification

Classify a **new observation** into one of the groups (Li et al., 2012)



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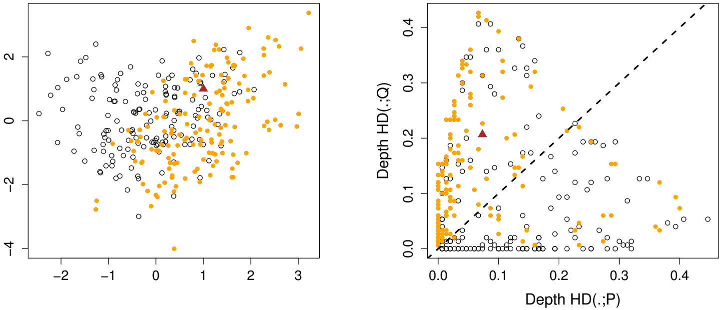
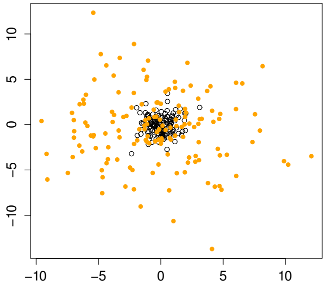
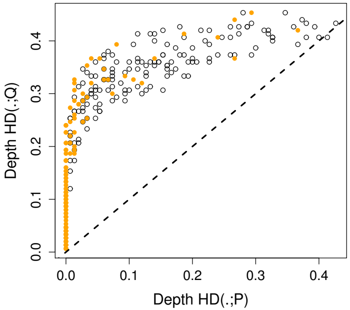


D-D Plots: Multivariate Q-Q Plots

D-D plots with **unequal scatters**





Computational Complexity of *hD*

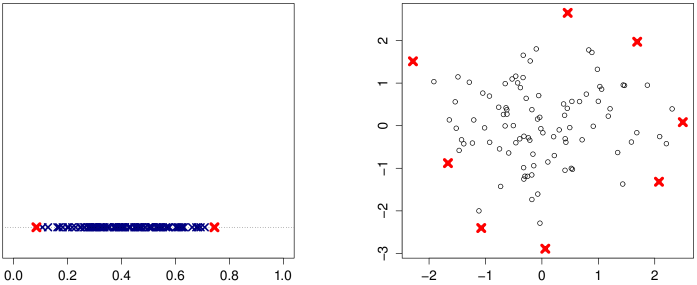
- best known exact algorithms have complexity $\mathcal{O}(\log(n)n^{d-1})$ (Rousseeuw and Struyf, 1998);
- **feasible computation** only for $n \leq 1000$ and $d \leq 5$;
- **approximations** of *hD* (Dyckerhoff, 2004)

$$hD(x;P) = \inf_{u \in \mathbb{S}^{d-1}} hD(\langle x, u\rangle; P_{\langle X, u\rangle}) \approx \min_{j=1,\dots,N} hD(\langle x, U_j\rangle; P_{\langle X, U_j\rangle}).$$

- choice of the parameter N and the distribution of U (Nagy, 2018+).

Ties

With increasing *d* the number of **depth-ties** increases



Some Open Problems

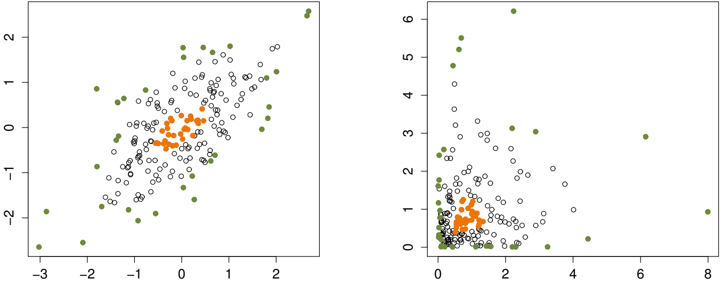
Little is known about

- **uniform** distributional asymptotics;
- higher order asymptotics;
- detection of **rough points**;
- finite/large sample **properties** of depth-based tests and estimators;
- **population depth** and its properties.

Simplicial Depth

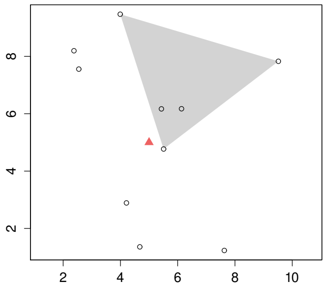
Simplicial depth (Liu, 1988) of an observation in $\mathbb{R}^d$

$$sD(x;P) = \mathsf{P}\left(x \in \mathbb{S}(X_1,\ldots,X_{d+1})\right).$$



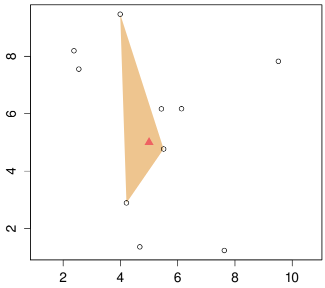
Simplicial Depth

$$sD(x;P_n) = \binom{n}{d+1}^{-1} \sum_{1 \leq X_{i_1} < \cdots < X_{i_{d+1}} \leq n} \mathbb{I}\left(x \in \mathbb{S}(X_{i_1},\ldots,X_{i_{d+1}})\right).$$



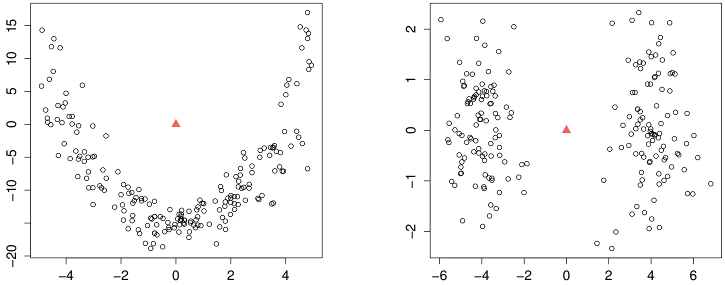
Simplicial Depth

$$sD(x;P_n) = \binom{n}{d+1}^{-1} \sum_{1 \leq X_{i_1} < \cdots < X_{i_{d+1}} \leq n} \mathbb{I}\left(x \in \mathbb{S}(X_{i_1},\ldots,X_{i_{d+1}})\right).$$



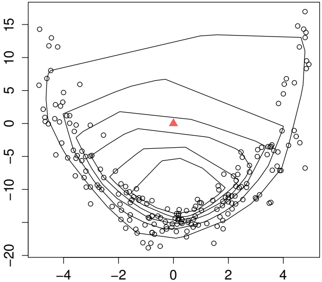
Unimodality / Quasi-Concavity

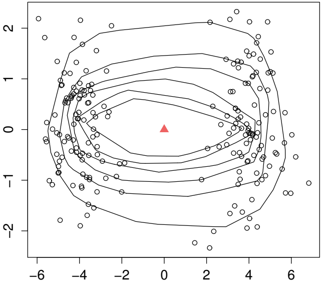
Proper depth is intended to be **unimodal** and **quasi-concave**



Unimodality / Quasi-Concavity

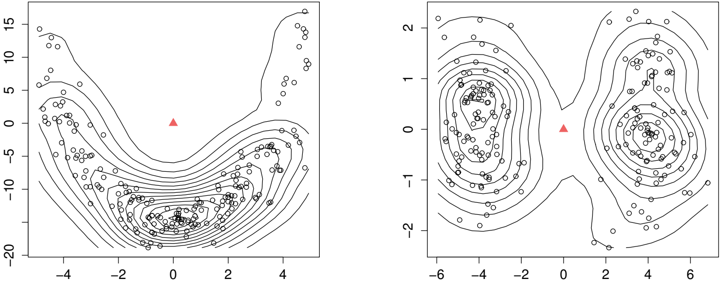
Proper depth is intended to be **unimodal** and **quasi-concave**





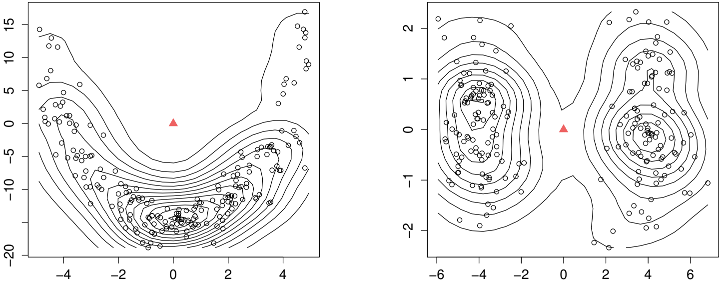
Local Depths

Relaxation of unimodality leads to **local depths**



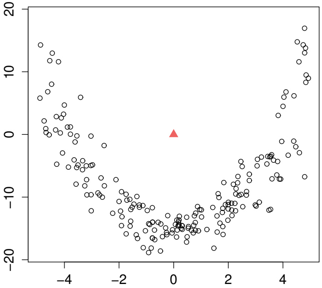
Likelihood Depth

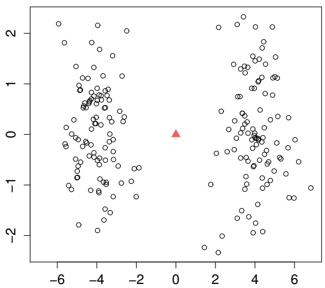
Multivariate **density estimator** (Fraiman and Meloche, 1999)



Local Halfspace Depth

Localization of *hD* (Paindaveine and Van Bever, 2013)





Local Halfspace Depth

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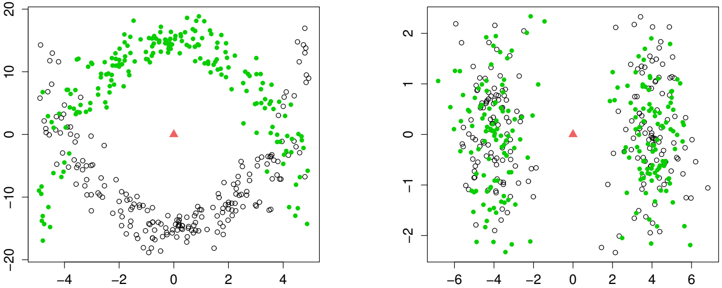
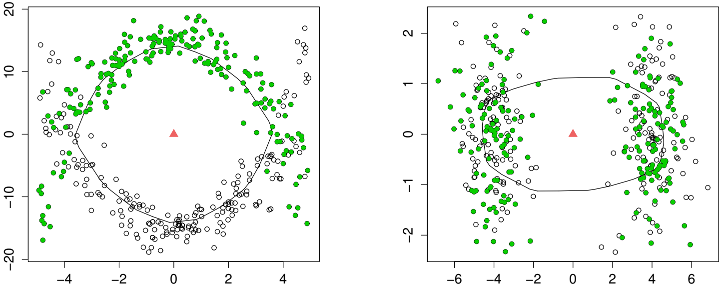


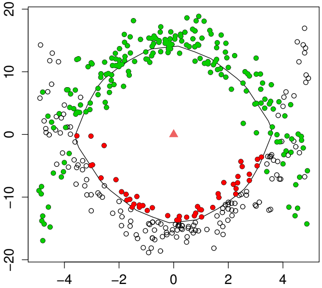
Local Halfspace Depth

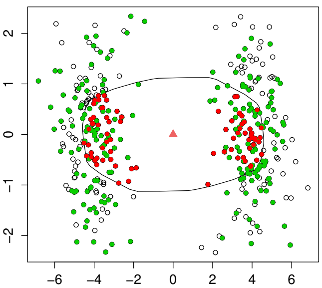
Localization of *hD* (Paindaveine and Van Bever, 2013)



Local Halfspace Depth

Localization of *hD* (Paindaveine and Van Bever, 2013)





Local Halfspace Depth

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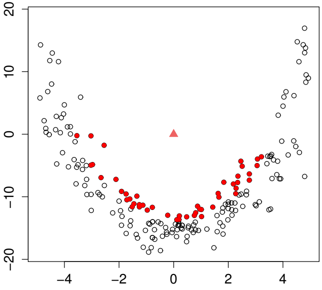

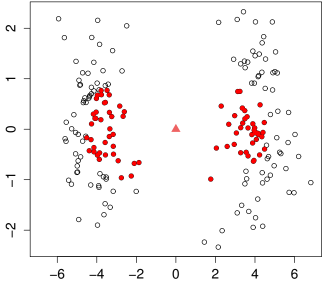

Local Halfspace Depth

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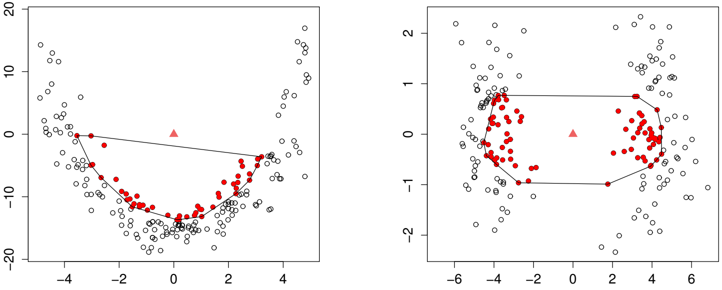


Further Extensions

Depths for more exotic data — **variants of the halfspace and simplicial depth**:

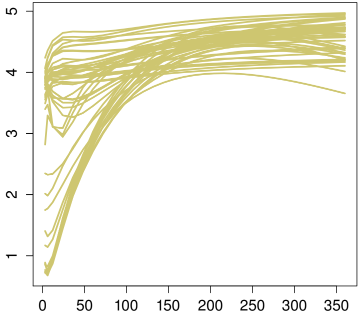
- for **directional data** (data in $\mathbb{S}^{d-1}$) (Liu and Singh, 1992);
- for data on **graphs and trees** (Small, 1997);
- for **infinite-dimensional** (functional) data (Fraiman and Muniz, 2001);
- for general **metric spaces** (Carrizosa, 1996);
- in **regression** problems (Rousseeuw and Hubert, 1999);
- ...

Many proposals, many tests, many simulations. **No sufficient comprehensive theory.**

Functional Data

$X \sim P \in \mathcal{P}(\mathcal{C})$ and $X_1, \ldots, X_n$ i.i.d. from P. Consider the depth of functional observations w.r.t. P (or P_n)

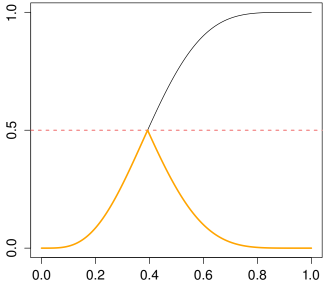
$$D \colon \mathcal{C} \times \mathcal{P}(\mathcal{C}) \to [0,1].$$

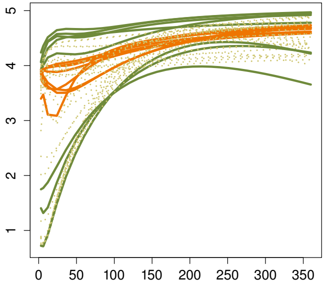


Integrated Depths

Fraiman and Muniz (2001), Nagy et al. (2016)

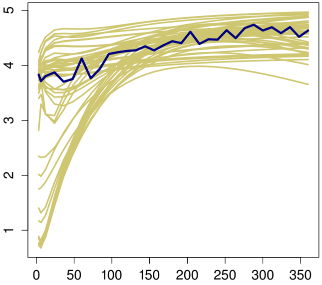
$$FD(x;P) = \int_0^1 hD_1(x(t), P_t)\,\mathrm{d}t, \qquad hD_1(u;Q) = 1/2 - |1/2 - \mathrm{F}_Q(u)|.$$





Data Contamination

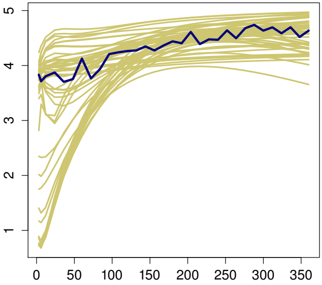
Consider **contaminated functional data**. Does the depth recognize the outlier?

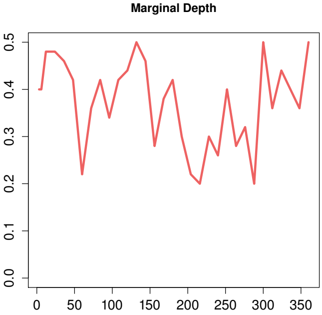


Data Contamination

Integrated depth (Fraiman and Muniz, 2001)

$$FD(x;P) = \int_0^1 hD(x(t);P_t)\,\mathrm{d}t$$

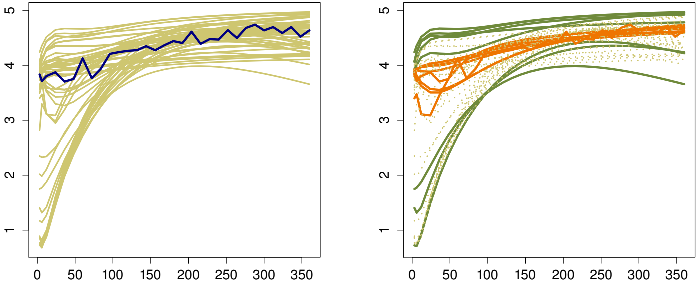




Integrated Depth

Integrated depth (Fraiman and Muniz, 2001)

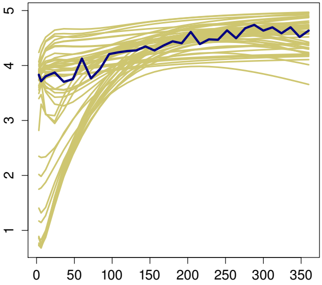
$$FD(x; P) = \int_0^1 hD(x(t); P_t) \, \mathrm{d}t$$

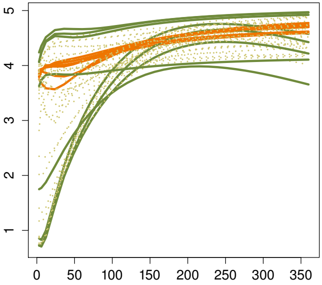


Depth with Derivatives

Integrated depth of differentiable functions (Hlubinka et al., 2015)

$$FD^{(2)}(x;P) = \int_0^1 hD\left(\left(x(t),x'(t)\right);\left(P_t,P'_t\right)\right)\,\mathrm{d}t$$

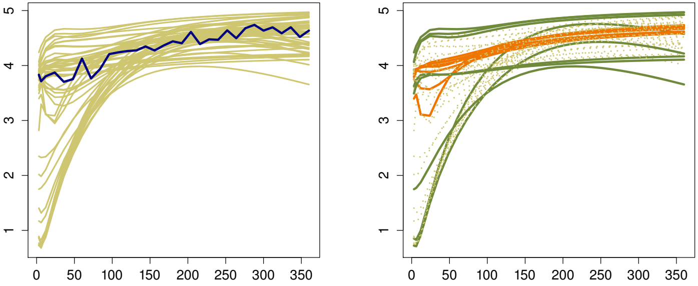




Depth with Derivatives without Derivatives

Second order integrated depth (Nagy et al., 2017)

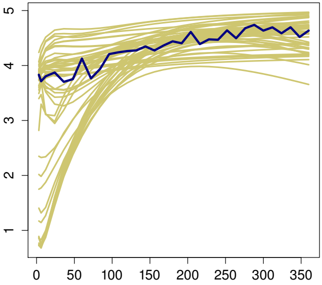
$$FD^2(x;P) = \int_0^1 \int_0^1 hD\left(\left(x(t), x(s)\right); \left(P_t, P_s\right)\right) \mathrm{d}t \, \mathrm{d}s$$

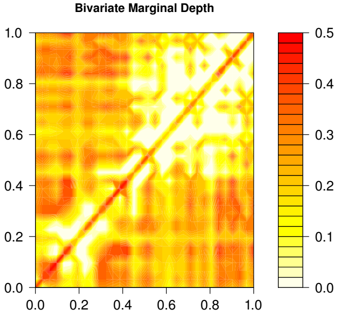


Higher Order Integrated Depth

Second order integrated depth (Nagy et al., 2017)

$$FD^2(x;P) = \int_0^1 \int_0^1 hD\left((x(t),x(s));(P_t,P_s)\right) \mathrm{d}t\,\mathrm{d}s$$





Higher Order Integrated Depth

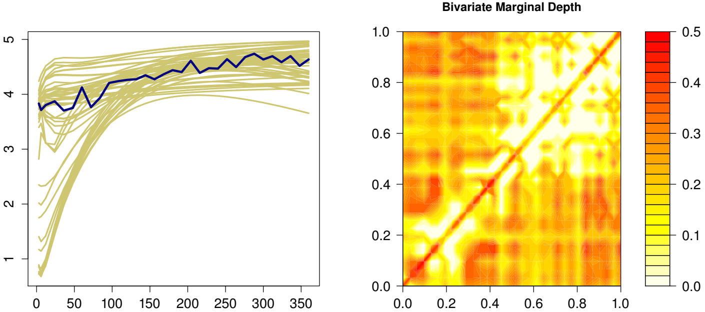
$$FD^{(2)}(x;P) = \int_0^1 hD\left(\left(x(t), x'(t)\right); \left(P_t, P_t'\right)\right) \mathrm{d}t$$

$$FD^{2}(x;P) = \int_0^1 \int_0^1 hD\left(\left(x(t), x(s)\right); \left(P_t, P_s\right)\right) \mathrm{d}t\, \mathrm{d}s$$

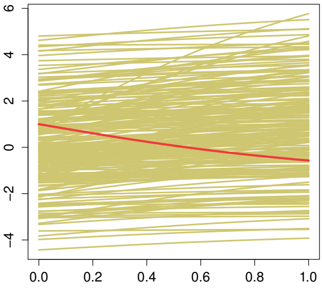
Theorem (Nagy et al., 2017)

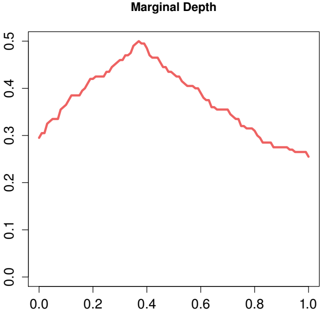
$$hD\left(\left(x(t), x'(t)\right); \left(P_t, P_t'\right)\right) = \lim_{h\to 0} hD\left(\left(x(t), x(t+h)\right); \left(P_t, P_{t+h}\right)\right).$$

Higher Order Integrated Depth

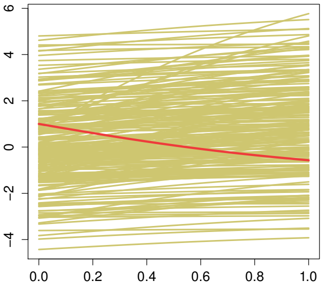


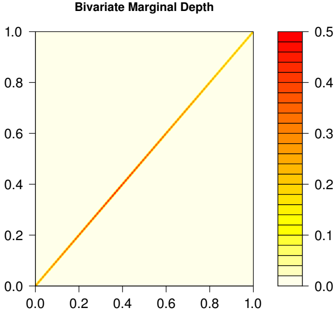
Higher Order Integrated Depth



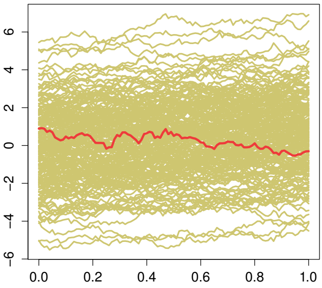


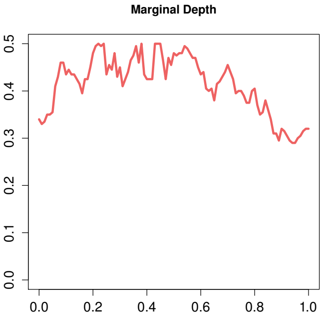
Higher Order Integrated Depth



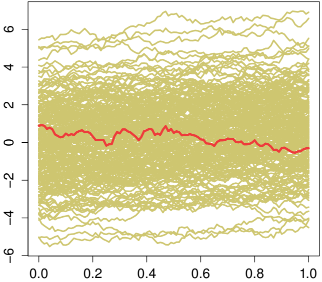


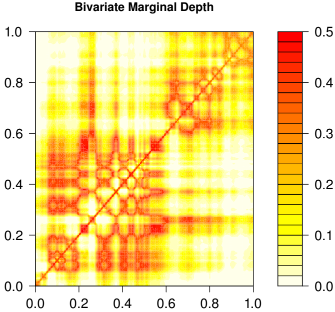
Higher Order Integrated Depth



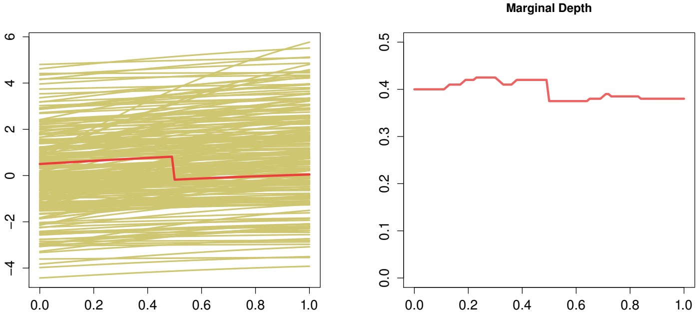


Higher Order Integrated Depth

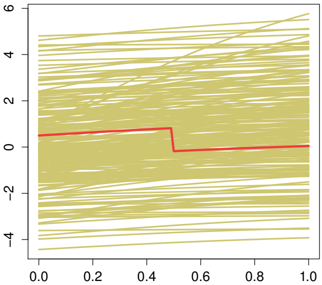


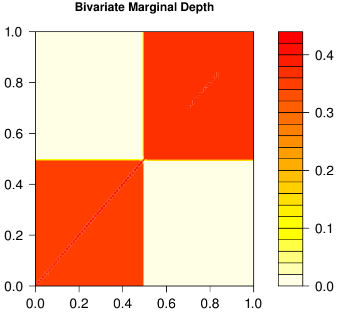


Higher Order Integrated Depth

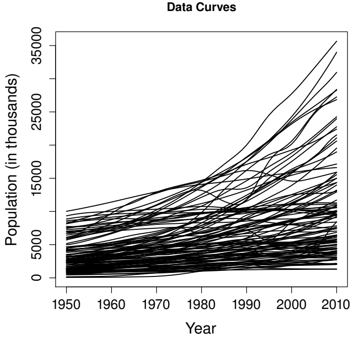


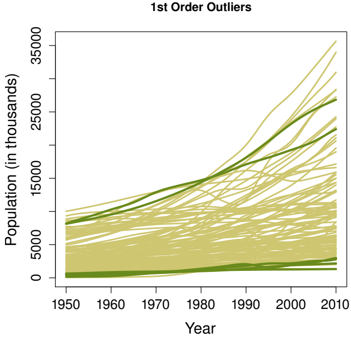
Higher Order Integrated Depth



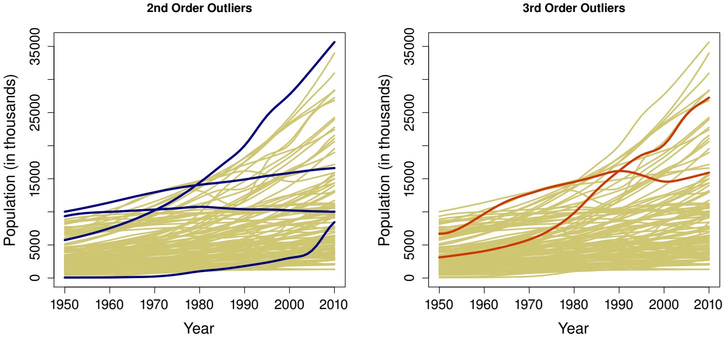


Example: World Population Growth

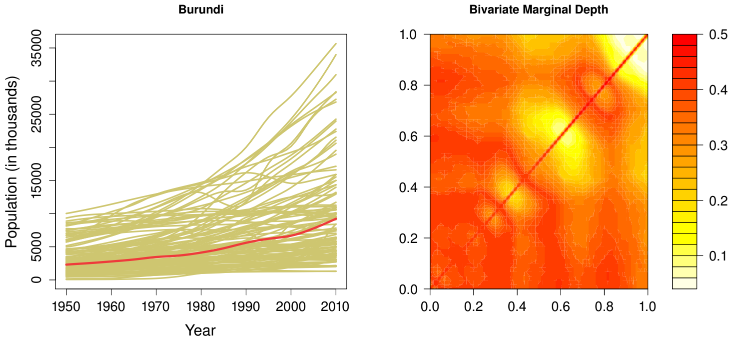




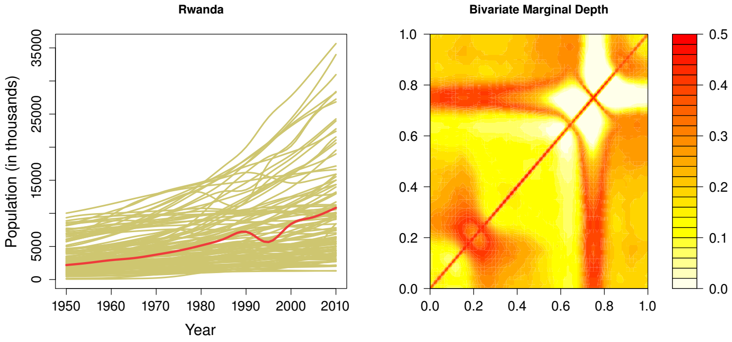
Atypical Curves of Higher Order



Burundi


Burundi
Population (in thousands)
0
5000
15000
25000
35000
1950
1960
1970
1980
1990
2000
2010
Year
Bivariate Marginal Depth
0.0
0.2
0.4
0.6
0.8
1.0
0.1
0.2
0.3
0.4
0.5

Rwanda


Rwanda
Population (in thousands)
0
5000
15000
25000
35000
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1960
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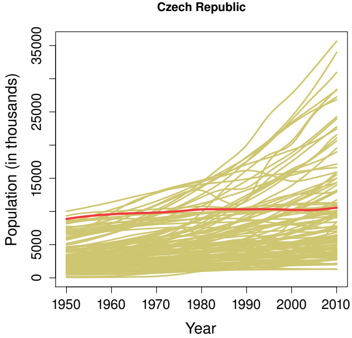
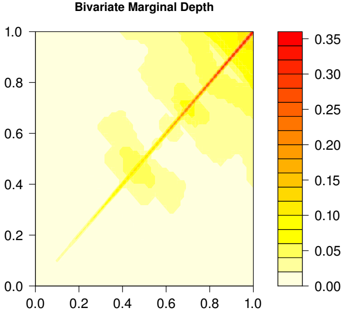
2000
2010
Year
Bivariate Marginal Depth
0.0
0.2
0.4
0.6
0.8
1.0
0.5
0.4
0.3
0.2
0.1
0.0

Czech Republic





Conclusions

Data depth is

- **easy** to understand (i.e. extremely popular);
- promises many **applications**; but also
- computationally intensive;
- with isolated and **underdeveloped theory**.

PRIMUS/17/SCI/03

Advanced Geometric Methods in Statistics

2018–2020

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Selected Literature

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