Floating bodies and approximation of convex bodies by polytopes

a convex body K in $\mathbb{R}^n$ is compact convex set with non-empty interior

a polytope P in $\mathbb{R}^n$ is the convex hull of finitely many points $x_1, \ldots, x_N$

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How well can a convex body be approximated by a polytope?

1. Approximation by a polytope P with

(i) a fixed number of vertices

(ii) a fixed number of facets = $(n-1)$-dimensional faces

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in (i) P is inscribed in K

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The symmetric difference metric

$$\begin{aligned}\Delta_v(K, L) &= \mathrm{vol}_n\Big((K \setminus L) \cup (L \setminus K)\Big) = |(K \setminus L) \cup (L \setminus K)| \\ &= |K \cup L| - |K \cap L|\end{aligned}$$

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When $K \subset L$,

$$\Delta_v(K,L) = |L| - |K|$$

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BEST APPROXIMATION

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$$= \frac{1}{2}\mathrm{del}_{n-1}\left(\int_{\partial K} \kappa(x)^{\frac{1}{n+1}}\, d\mu(x)\right)^{\frac{n+1}{n-1}}$$

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del_{n-1} is a constant that depends only on n

Best Approximation for N large

$$\begin{aligned}\Delta_v(K, P_{\text{best}}) &= |K| - |P_{\text{best}}| \\ &\sim \frac{1}{2}\text{del}_{n-1}\left(\int_{\partial K}\kappa(x)^{\frac{1}{n+1}}d\mu(x)\right)^{\frac{n+1}{n-1}}\frac{1}{N^{\frac{2}{n-1}}}\end{aligned}$$

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There is a numerical constant $c>0$ such that

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Affine surface area appears

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- there is an affine isoperimetric inequality

$$\left(\frac{as(K)}{as(B_2^n)} \right)^{n+1} \leq \left(\frac{|K|}{|B_2^n|} \right)^{n-1},$$

with equality iff K is an ellipsoid (Blaschke; Petty)

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- PDEs (Lutwak+Oliker, Trudinger, Wang, etc.)

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Then: Leichtweiss, Lutwak, Meyer+Werner, Schütt+Werner, Werner,.......

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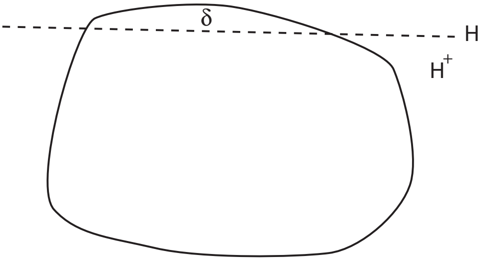
The (convex) **floating body** is the intersection of all halfspaces H^+ whose defining hyperplanes H cut off a set of volume δ of K.

$$K_\delta = \bigcap_{|H^- \cap K| = \delta} H^+$$



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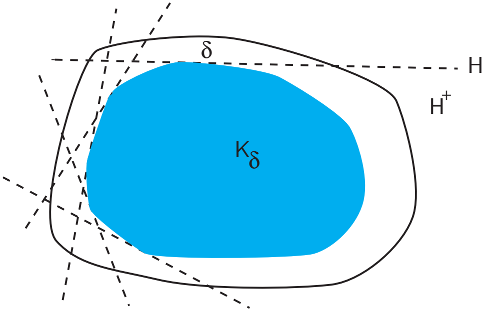

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Other application of the floating body: Data depth

Random approximation

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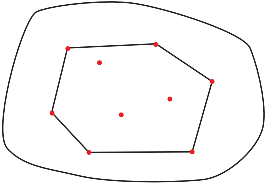
when chosen in K, not ALL points become vertices



More generally:

$f : \partial K \to \mathbb{R}^+$ strictly positive a.e. on ∂K, $\int_{\partial K} f d\mu = 1$

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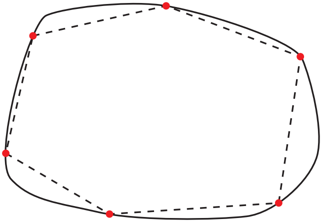
Choose N points $x_1, \ldots x_N$ w.r.t. to $\mathbb{P}_f$ on ∂K

As before we call their convex hull $[x_1, \ldots, x_N]$ a **random polytope**. **Every** point chosen becomes a vertex



The **expected volume** of such a random polytope is

$$\mathbb{E}_N(\partial K, \mathbb{P}_f) = \int_{\partial K} \cdots \int_{\partial K} \Big| [x_1, \ldots, x_N] \Big| d\mathbb{P}_f(x_1) \ldots d\mathbb{P}_f(x_N)$$

Theorem (Schütt&Werner)

Let K be a convex body in $\mathbb{R}^n$ with sufficiently regular boundary.

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- $$c_n = \frac{(n-1)^{\frac{n+1}{n-1}} \Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}}$$

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How do best and random approximation compare?

- $|K| - |P_{\text{best}}| \sim \left(\frac{1}{N}\right)^{\frac{2}{n-1}} \left(\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu(x)\right)^{\frac{n+1}{n-1}} \frac{1}{2}\mathrm{del}_{n-1}$
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With an absolute constant c

$$\frac{1}{2}\mathrm{del}_{n-1} \leq c_n \leq \left(1 + \frac{c \log n}{n}\right) \frac{1}{2}\mathrm{del}_{n-1}$$

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These restrictions need to be dropped

Again: we concentrate on the vertex case

Drop requirement that $P_N \subset K$

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Theorem (Ludwig, Schütt, Werner)

Let K be C^2_+. There is a constant $c > 0$ s.th. for all N large enough there is a polytope P in $\mathbb{R}^n$ with at most N vertices s.th.

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- When $P \subset K$ (Bronsteyn&Ivanov)

$$\Delta_v(K,P) = |K| - |P| \leq c\ n\,|K| \left(\frac{1}{N}\right)^{\frac{2}{n-1}}$$

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Theorem (Ludwig, Schütt, Werner)

Let K be C^2_+. There is a constant $c > 0$ s.th. for all N large enough there is a polytope P in $\mathbb{R}^n$ with at most N vertices s.th.

$$\Delta_v(K, P) \leq c\ |K|\ \left(\frac{1}{N}\right)^{\frac{2}{n-1}}$$

- the corresponding result for facets holds as well
- When $P \subset K$ (Bronsteyn&Ivanov)

$$\Delta_v(K, P) = |K| - |P| \leq c\ n\ |K|\ \left(\frac{1}{N}\right)^{\frac{2}{n-1}}$$

- If we drop the restriction, we gain by a factor of dimension: n

When $P \subset K$, we actually have

$$c_1 \; n \; |K| \; \left(\frac{1}{N}\right)^{\frac{2}{n-1}} \leq \Delta_v(K,P) = |K| - |P| \leq c_2 \; n \; |K| \; \left(\frac{1}{N}\right)^{\frac{2}{n-1}}$$

Upper bound: Bronsteyn & Ivanov
Lower bound: Gordon, Reisner & Schütt

- lower bound in the general case

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Theorem (Böröczky)

For every polytope P_N with at most N vertices

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GAP between lower and upper bound by a factor of dimension

Theorem (Grote+Werner)

Let K be a convex body in $\mathbb{R}^n$ that is C^2_+. Let $f : \partial K \to \mathbb{R}_+$ be a continuous and strictly positive function with

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$$\int_{\partial K} f(x) d\mu(x) = 1.$$

Then, for N sufficiently large, there exists a polytope P_f in $\mathbb{R}^n$ having N vertices such that

$$\Delta_v(K, P_f) \le a\, N^{-\frac{2}{n-1}} \int_{\partial K} \frac{\kappa(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} d\mu(x),$$

where $a \in (0, \infty)$ is an absolute constant.

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Comments

- The proof is via a random construction:

We choose at random points $x_1, \dots, x_N$ with respect to $\mathbb{P}_f = f\mu$ on ∂K and approximate $(1-c)\partial K$ and re-adjust

$$\Delta_v(K, P_f) \leq a\, N^{-\frac{2}{n-1}} \int_{\partial K} \frac{\kappa(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} d\mu(x)$$

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• One gains by a factor of dimension if one allows arbitrarily positioned polytopes

$$|K| - \mathbb{E}_N(\partial K, \mathbb{P}_f) \;\sim\; n\; N^{-\frac{2}{n-1}} \int_{\partial K} \frac{\kappa(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} d\mu(x)$$

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- Assumption C^2_+ can be relaxed

$$\Delta_v(K, P_f) \leq a\, N^{-\frac{2}{n-1}} \int_{\partial K} \frac{\kappa(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} d\mu(x)$$

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• Minimum on RHS is attained for the normalized affine surface area measure with density

$$f_1(x) = \frac{\kappa(x)^{\frac{1}{n+1}}}{as(K)}$$

where

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$$\begin{aligned}\Delta_v(K, P_{f_1}) &\leq a\, N^{-\frac{2}{n-1}}\ as(K)^{\frac{n+1}{n-1}} \\ &\leq a\, N^{-\frac{2}{n-1}}\, \frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(B_2^n)}\ as(B_2^n)^{\frac{n+1}{n-1}}\end{aligned}$$

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&\leq a\, N^{-\frac{2}{n-1}}\, \mathrm{vol}_n(K)
\end{aligned}$$

The latter is the above mentioned result by
Ludwig+Schütt+Werner: $\Delta_v(K, P) \leq c\ N^{-\frac{2}{n-1}}\ \mathrm{vol}_n(K)$