Statistical depth:
Part II: Depth in mathematics

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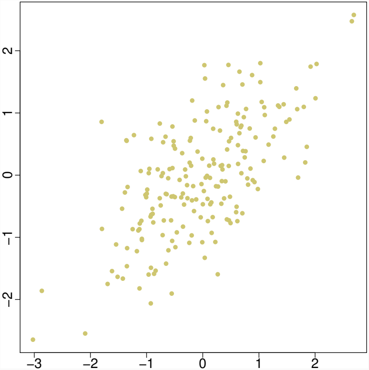
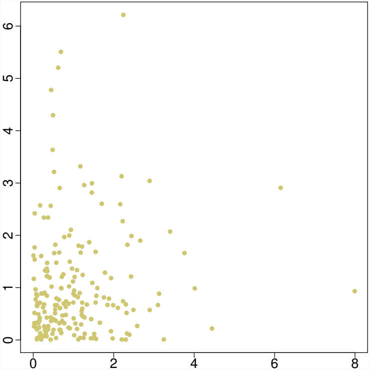
Consider the depth of $x \in \mathbb{R}^d$ w.r.t. $P \in \mathcal{P}\left(\mathbb{R}^d\right)$

$$D\colon \mathbb{R}^d \times \mathcal{P}\left(\mathbb{R}^d\right) \to [0,1]\colon (x,P) \mapsto D(x;P).$$





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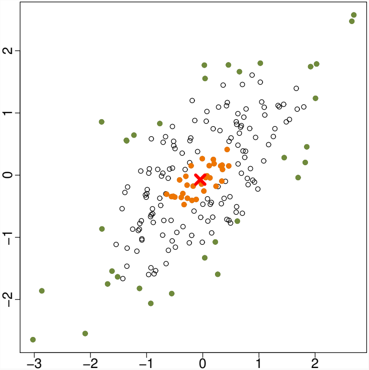
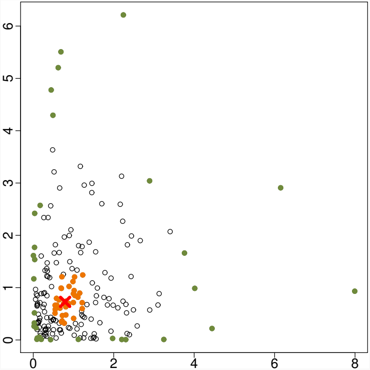
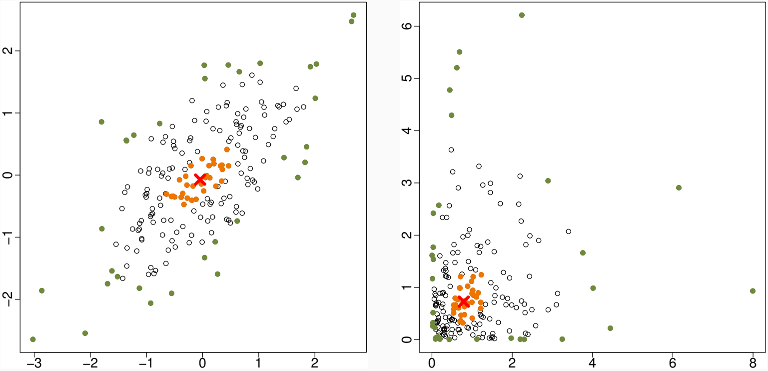




Halfspace depth (Tukey, 1975) of $x \in \mathbb{R}^d$

$$hD(x;P) = \inf_{H \in \mathcal{H}(x)} P(H).$$



Statistical depth is a function

$$D\colon \mathbb{R}^d \times \mathcal{P}\left(\mathbb{R}^d\right) \to [0,1]\colon (x,P) \mapsto D(x;P),$$

that satisfies (Zuo and Serfling, 2000b)

1. affine invariance;
2. maximality at the centre for symmetric distributions;
3. monotonicity relative to the depth median;
4. vanishing at infinity.

Sometimes it is required also (Serfling, 2006b)

5. upper semi-continuity as a function of x;
6. continuity as a functional of P;
7. quasi-concavity in x.

Symmetry of random variables

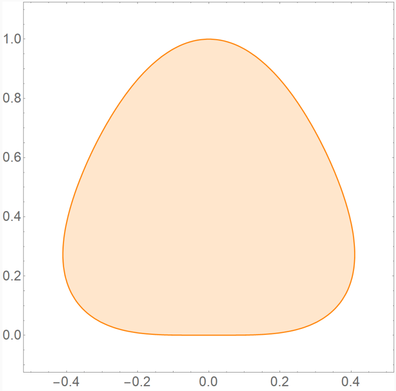
- Depth of a median — Grünbaum's theorem
- Measures of symmetry
- Funk's characterization of symmetry

Quasi-Concavity: Floating body

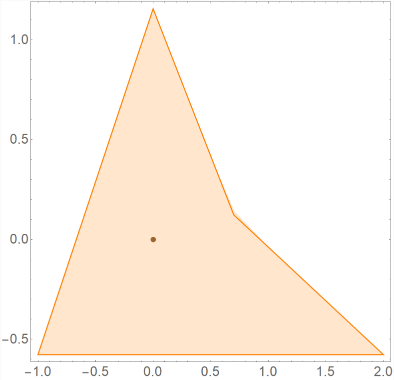
- Dupin's floating body
- Convex floating body

Symmetry of random variables

Convex body is a non-empty, compact and convex set $K \subset \mathbb{R}^d$. We write also $K \in \mathcal{K}^d$ (Webster, 1994; Schneider, 2014).

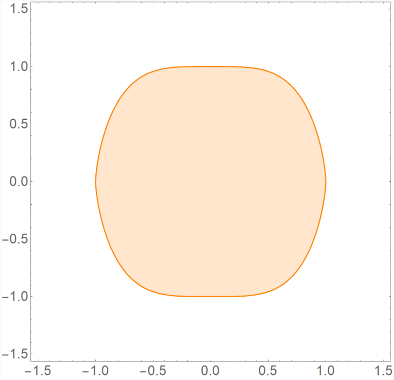


Star body is $K \subset \mathbb{R}^d$, such that for some $x \in K$ and any $k \in K$ it holds $[x, k] \subset K$. (Schneider, 2014; Groemer 1996).



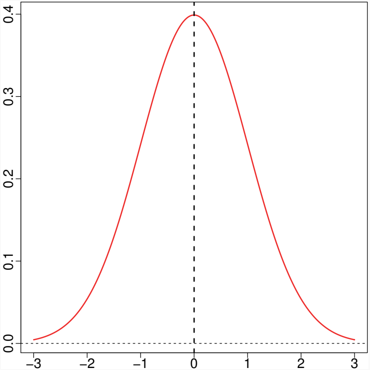
A convex body $K \in \mathcal{K}^d$ is (centrally) symmetric about $\theta \in \mathbb{R}^d$ iff

$$K - \theta = -(K - \theta).$$



$X \sim P \in \mathcal{P}(\mathbb{R})$ is (centrally) symmetric about $\theta \in \mathbb{R}$ iff

$$X - \theta \stackrel{d}{=} -(X - \theta).$$



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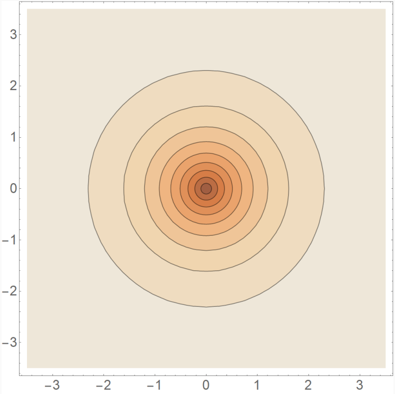
Multiple generalizations to $\mathcal{P}\left(\mathbb{R}^d\right)$ (Serfling, 2006):

- spherical symmetry;
- elliptical symmetry;
- central symmetry;
- angular symmetry (Liu, 1988);
- halfspace symmetry (Zuo and Serfling, 2000).

$X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is spherically symmetric about $\theta \in \mathbb{R}^d$ iff

$$X - \theta \stackrel{d}{=} A(X - \theta)$$

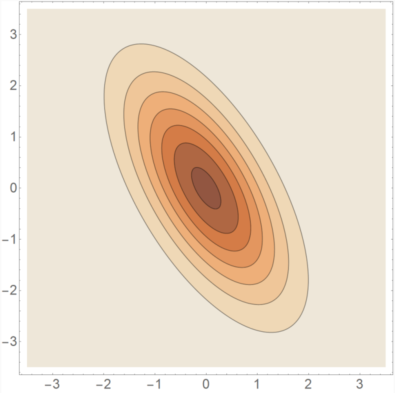
for any $A \in \mathbb{R}^{d \times d}$ orthogonal.



$X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is elliptically symmetric about $\theta \in \mathbb{R}^d$ iff

$$X \stackrel{d}{=} A^\mathsf{T} Y + \theta$$

for $Y \in \mathbb{R}^k$ spherically symmetric, and $A \in \mathbb{R}^{k \times d}$ of rank k $(\leq d)$.



$X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is centrally symmetric about $\theta \in \mathbb{R}^d$ iff

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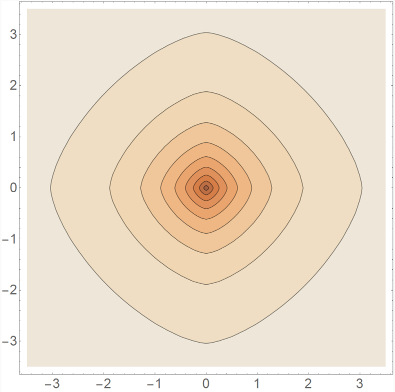
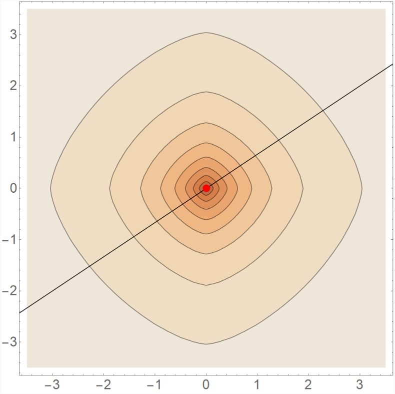


$X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is centrally symmetric about $\theta \in \mathbb{R}^d$ iff

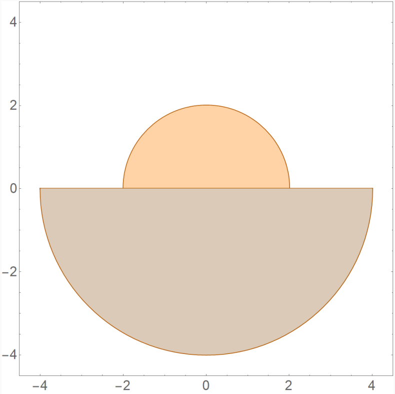
$$\langle X - \theta, u \rangle \text{ are (centrally) symmetric for all } u \in \mathbb{S}^{d-1}.$$



Angular symmetry

$X \sim P \in \mathcal{P}(\mathbb{R}^d)$ is angularly symmetric about $\theta \in \mathbb{R}^d$ iff (Liu, 1988)

$$\frac{X-\theta}{\|X-\theta\|} \stackrel{d}{=} -\frac{X-\theta}{\|X-\theta\|}. \qquad \text{(here } 0/0 = 0\text{)}$$



Angular symmetry

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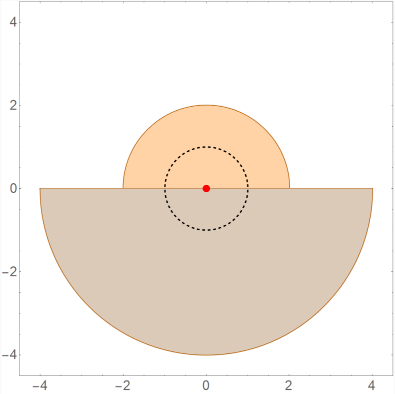
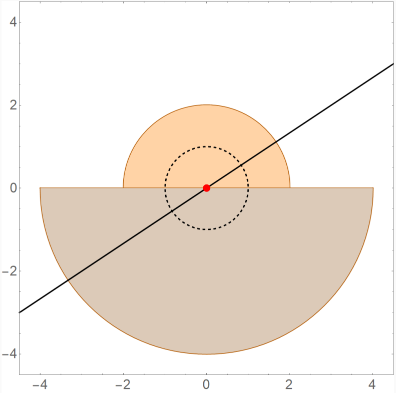
$$\frac{X-\theta}{\|X-\theta\|} \text{ is centrally symmetric about 0.}$$



Halfspace symmetry

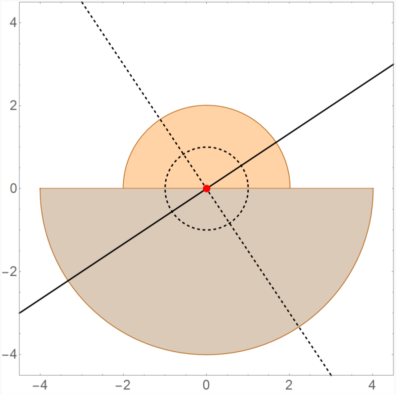
$X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is halfspace symmetric about $\theta \in \mathbb{R}^d$ iff (Zuo and Serfling, 2000)

$$hD(\theta; P) \geq 1/2.$$



$X \sim P \in \mathcal{P}(\mathbb{R}^d)$ is halfspace symmetric about $\theta \in \mathbb{R}^d$ iff

$$\langle \theta, u \rangle \text{ is a median of } \langle X, u \rangle \text{ for all } u \in \mathbb{S}^{d-1}.$$

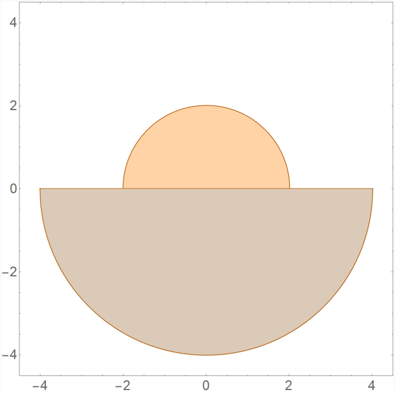


Proposition (Zuo and Serfling, 2000)
In the space of probability measures $\mathcal{P}\left(\mathbb{R}^d\right)$

$$\begin{aligned}\textit{spherical symmetry} &\implies \textit{elliptical symmetry} \implies \textit{central symmetry} \\ &\implies \textit{angular symmetry} \implies \textit{halfspace symmetry.}\end{aligned}$$

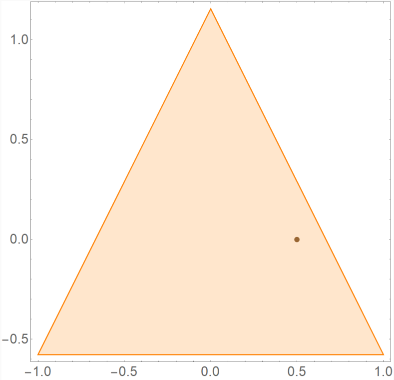
No implication can be reversed.

central symmetry $\not\Longleftarrow$ angular symmetry

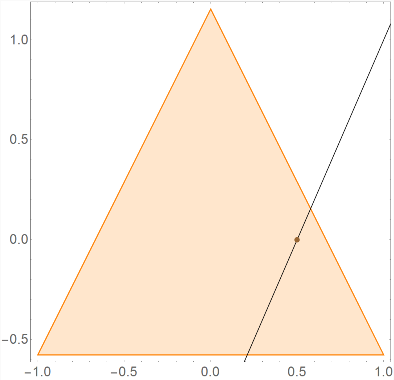


Central $\implies$ angular $\implies$ halfspace symmetry

angular symmetry $\not\Longleftarrow$ halfspace symmetry



angular symmetry $\not\Longleftarrow$ halfspace symmetry



Proposition (Zuo and Serfling, 2000, Theorem 2.6)
Suppose a random vector X is halfspace symmetric about a unique point $\theta \in \mathbb{R}^d$, end either

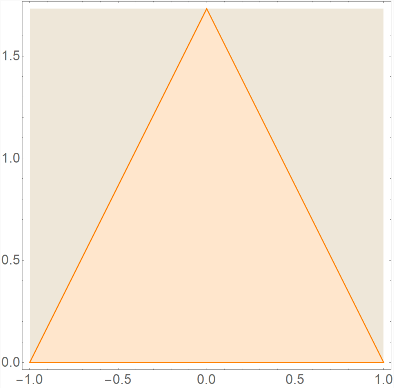
1. *X is absolutely continuous, or*
2. *X is discrete and $\mathrm{P}(X = \theta) = 0$.*

Then X is angularly symmetric about θ.

Remark. The centre of halfspace symmetry of $X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is a unique point, unless $d = 1$ and X has two medians.

Distribution which is not halfspace symmetric:

$$\sup_{x \in \mathbb{R}^2} hD(x; P) = 4/9$$



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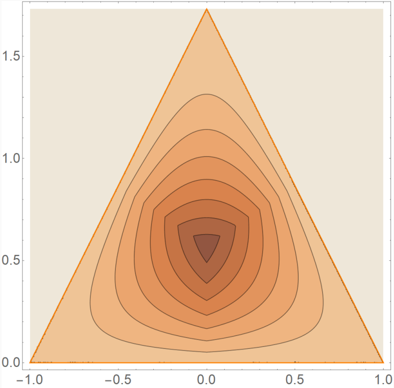

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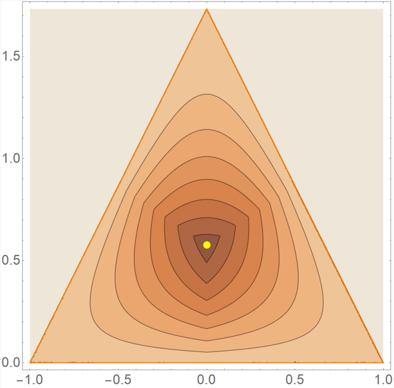


H-symmetry $\supsetneq$ A-symmetry $\supsetneq$ C-symmetry $\supsetneq$ E-symmetry $\supsetneq$ S-symmetry

A desired property of the data depth:

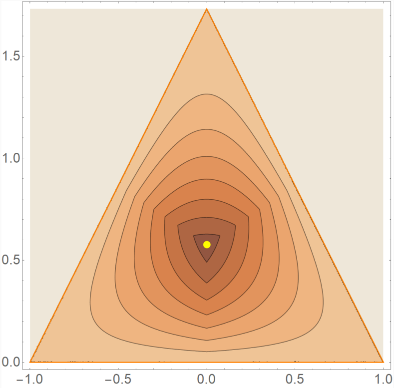
2. maximality at the centre for symmetric distributions;

Proposition (Zuo and Serfling, 2000b)
For $X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ symmetric about $\theta \in \mathbb{R}^d$

$$hD\,(\theta; P) = \sup_{x \in \mathbb{R}^d} hD(x; P).$$

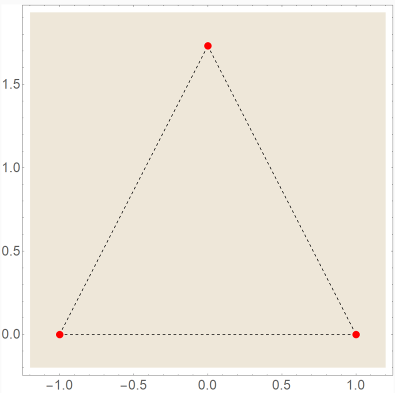
Testing for H-symmetry of $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ (Dutta et al., 2011)

$$T_n = \left(1/2 - \sup_{x \in \mathbb{R}^d} hD(x; P_n)\right)_+$$



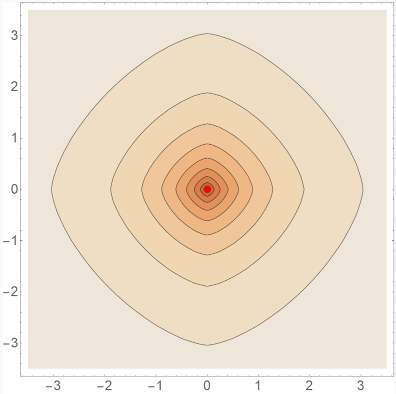
For $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ uniform in the vertices of a simplex (Donoho and Gasko, 1992)

$$\sup_{x \in \mathbb{R}^d} hD(x;P) = (d+1)^{-1} \xrightarrow[d \to \infty]{} 0$$



For $X \sim P$ angularly symmetric about $\theta \in \mathbb{R}^d$
(Rousseeuw and Struyf, 2004, Theorem 1)

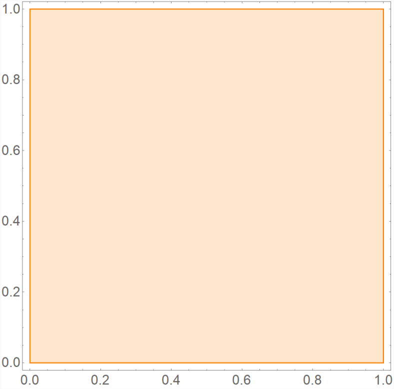
$$\sup_{x \in \mathbb{R}^d} hD(x; P) = hD(\theta, P) = 1/2 + P(\{\theta\})/2.$$

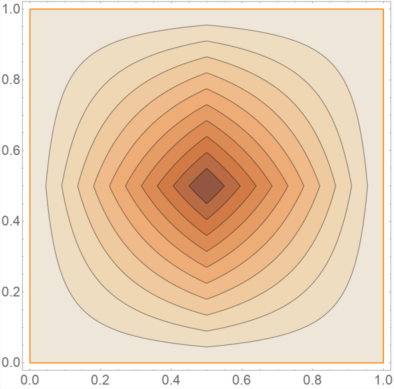


Problem (Donoho and Gasko, 1992; Dutta et al., 2011)

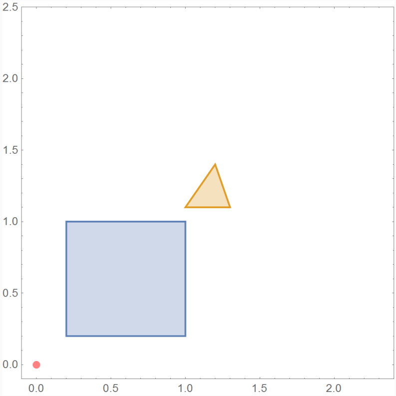
The depth of a median of an absolutely continuous distribution in $\mathbb{R}^d$ lies in the interval $[1/(d+1), 1/2]$. Can we say more?

Population depth of a convex body $K \in \mathcal{K}^d$





$$K + L = \{x + y \colon x \in K, y \in L\}\,, \quad \lambda K = \{\lambda x \colon x \in K\}$$



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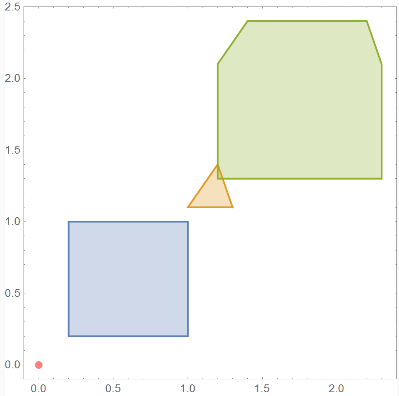

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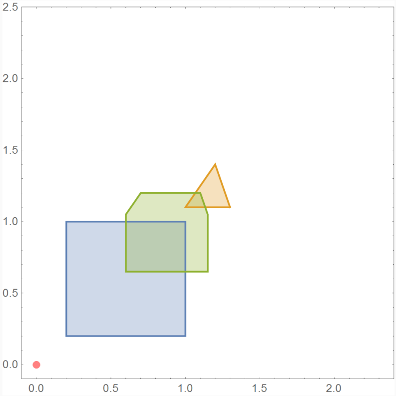

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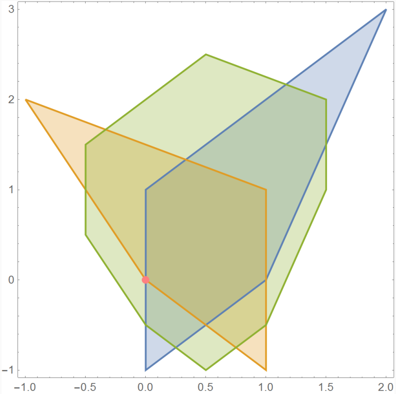

$$K+L=\{x+y\colon x\in K, y\in L\}\,,\quad \lambda K=\{\lambda x\colon x\in K\}$$

Proposition (Brunn, 1887; Minkowski, 1896)
Let $K,L\subset\mathbb{R}^d$ be convex bodies, $\operatorname{vol}(K)=\operatorname{vol}(L)=1$. *Then* $\operatorname{vol}((K+L)/2)\geq 1$, *with equality iff K is a translate of L.*

- Function $K\mapsto\operatorname{vol}(K)^{1/d}$ is concave on $\mathcal{K}^d$.

Proposition (Grünbaum, 1960)
Let $K \in \mathcal{K}^d$, $\mathrm{vol}(K) = 1$. *Then there is a point* $x \in K$ *such that*

$$hD(x; K) \geq \left(\frac{d}{d+1}\right)^d.$$

The bound is attained iff K *is a simplex.*

Proposition (Grünbaum, 1960)
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The bound is attained iff K is a simplex.

- Grübaum proves a stronger statement:
  The theorem holds with $x = \mathrm{E}\, K$.
- $\lim_{d\to\infty} \left(\frac{d}{d+1}\right)^d = \exp(-1) \approx 0.37$.

Proposition (Winternitz, 1917)
For $K \in \mathcal{K}^2$ with centroid $x \in K$

$$hD(x;K) \geq 4/9 = \left(\frac{2}{2+1}\right)^2.$$

This bound is attained iff K is a triangle.

Arthur Winternitz (1893 – 1961)

- graduated (1917) and worked (1917 – 1939) at the German University in Prague;
- the Winternitz theorem first appears in Blaschke (1923);
- independently rediscovered by
  - 1935 Lavrentjev and Lyusternik;
  - 1945 Neumann;
  - 1951 Yaglom and Boltyanskii;
  - 1955 Ehrhart;
  - 1958 Newman;
- Theorem extended to $d = 3$ by Ehrhart (1956);
- For general d conjectured by Ehrhart (1955), proved independently by Grünbaum (1960) and Hammer (1960).

Version with $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ (Donoho and Gasko, 1992):

$$\sup_{x \in \mathbb{R}^d} hD(x; P) \geq \frac{1}{d+1}.$$

Previously noted by

- Neumann (1955), Yaglom and Boltyanskii (1951), Newman (1958) for $d = 2$;
- Rado (1946), Birch (1959), Grünbaum (1960) for all d;
- Grünbaum (1960) shows that the bound is attained iff P is uniform in the vertices of a simplex.

Definition (Borell, 1974)
For $\kappa \in [-\infty, \infty)$ we say that $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is a κ-concave measure iff

$$P(\lambda A + (1-\lambda)B) \geq \begin{cases} P(A)^\lambda P(B)^{1-\lambda} & \text{for } \kappa = 0, \\ \min\{P(A), P(B)\} & \text{for } \kappa = -\infty, \\ (\lambda P(A)^\kappa + (1-\lambda)P(B)^\kappa)^{1/\kappa} & \text{otherwise.} \end{cases}$$

for all $A, B \subset \mathbb{R}^d$ Borel and $\lambda \in [0, 1]$.

For a κ-concave measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$:

- for any $\tau < \kappa$ is P also τ-concave;
- if $\kappa > 1$, P must be a Dirac measure;
- uniform measures on convex bodies are $1/d$-concave;
- if P has a density, then $\kappa \leq 1/d$;
- if $\kappa = 0$, P is called log-concave;
- if $\kappa > -1$, then P has a mean value;
- if $\kappa = -\infty$, P is called quasi-concave.

Proposition (Bobkov, 2010, Theorem 5.2)
For $\kappa \in (-1, 1]$ and κ-concave $X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$

$$hD(\mathsf{E}\,X; P) \geq \begin{cases} \exp(-1) & \textit{for } \kappa = 0, \\ \left(\frac{1}{1+\kappa}\right)^{1/\kappa} & \textit{otherwise.} \end{cases}$$

There are κ-concave measures that attain this bound.

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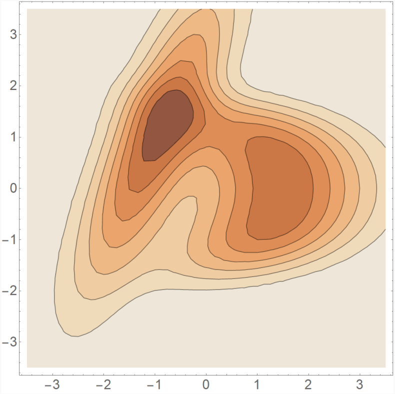
There are κ-concave measures that attain this bound.

Problem

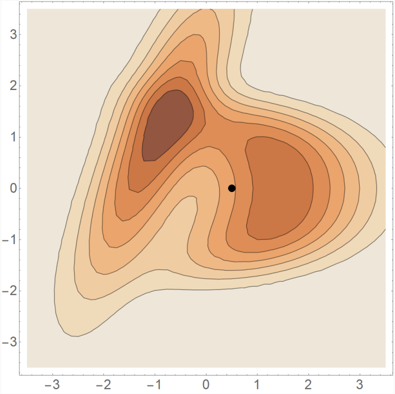
Can we say something about the case $\kappa \leq -1$?
What about points other than $\mathrm{E}\,X$?

Note: For $\kappa = 0$ the theorem was already known in economics.
(Caplin and Nalebuff, 1988)

Optimal shop location problem (Carrizosa, 1996)



Optimal shop location problem (Carrizosa, 1996)



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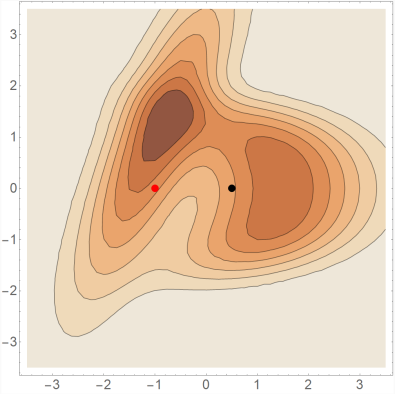

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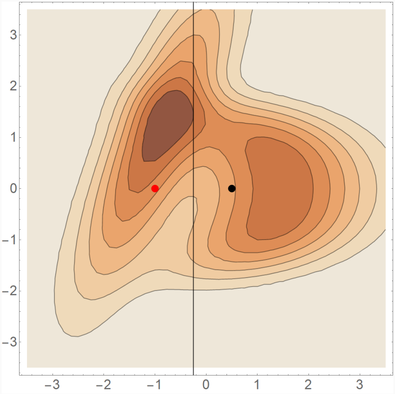

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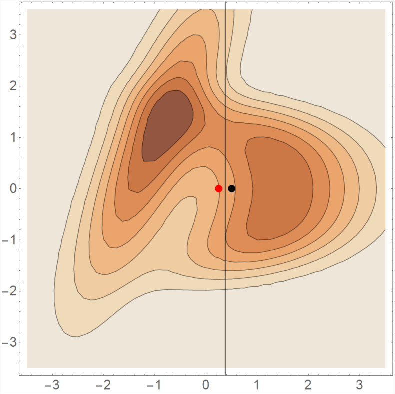

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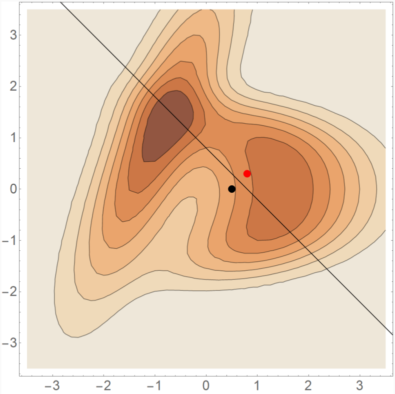

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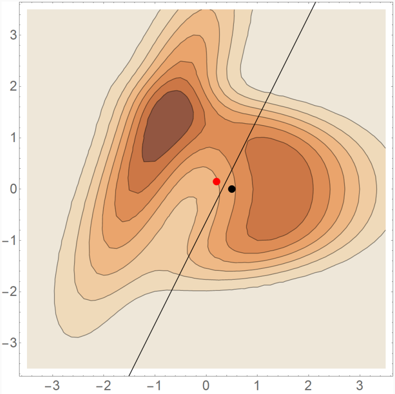

Proposition (Winternitz, 1917)
For $K \in \mathcal{K}^2$ with centroid $x \in K$

$$hD(x; K) \geq 4/9 = \left(\frac{2}{2+1}\right)^2 .$$

This bound is attained iff K is a triangle.

Definition (Winternitz, 1917; Blaschke, 1923)
For $K \in \mathcal{K}^d$, $x \in K$ and a halfspace $H \in \mathcal{H}(x)$, consider

$$f(H,x) = \frac{\operatorname{vol}(K \cap H)}{\operatorname{vol}(K) - \operatorname{vol}(K \cap H)}$$

and $f(x) = \min\{f(H,x) \colon H \in \mathcal{H}(x)\}$. The Winternitz measure of symmetry of the body K is then given by

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$$f(x) = \frac{hD(x; K)}{1 - hD(x; K)}$$

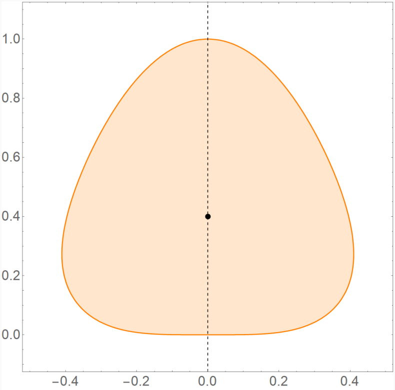
Definition (Grünbaum, 1963)
A function $s\colon \mathcal{K}^d \to [0,1]$ is called a measure of symmetry iff

1. $s(K) = 1$ iff K has a centre of (central) symmetry;
2. $s(K) = s(T(K))$ for every $K \in \mathcal{K}^d$ and every non-singular affine transformation $T\colon \mathbb{R}^d \to \mathbb{R}^d$;
3. s is continuous on $\mathcal{K}^d$.

For $K \in \mathcal{K}^d$, $x \in K$

$$D(x;K) = \inf_{H \in \mathcal{H}(x)} \frac{\mathrm{dist}(\partial H, \partial H_1)}{\mathrm{dist}(\partial H, \partial H_2)},$$

where H_1 and H_2 and parallel to H, and support K.



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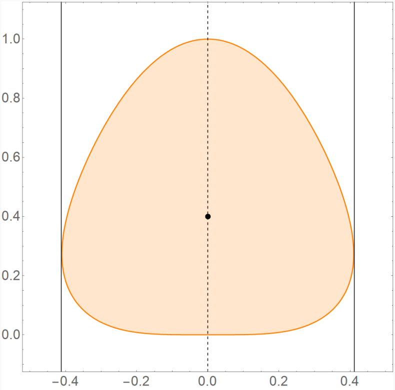

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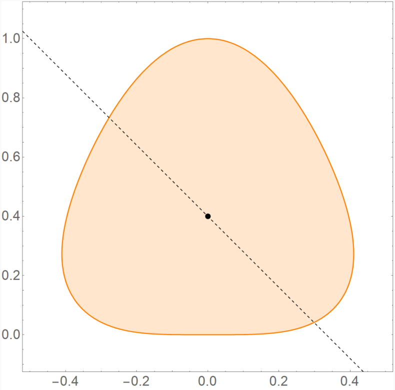

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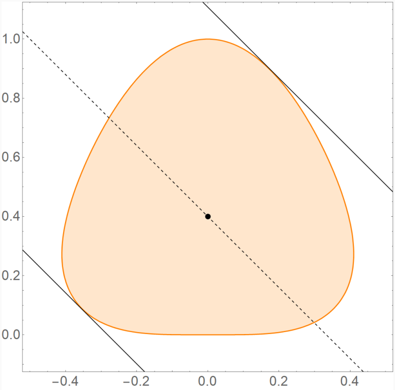

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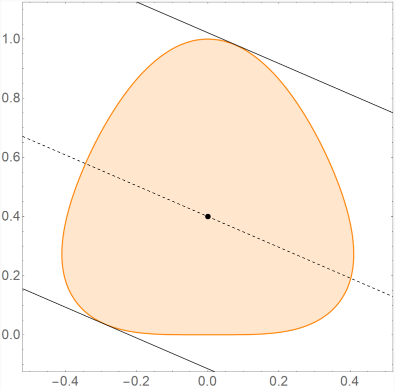
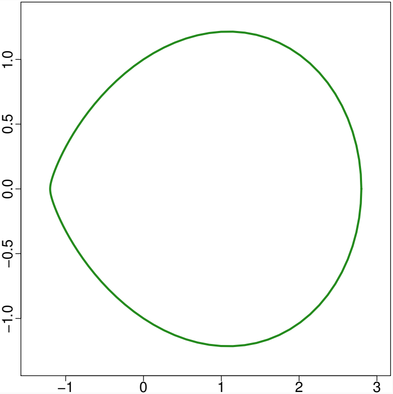


Kövner-Besicovitch measure of symmetry

For $K \in \mathcal{K}^d$, $x \in K$ (Besicovitch, 1951)

$$D(x; K) = \text{vol}\left(K \cap (2x - K)\right).$$



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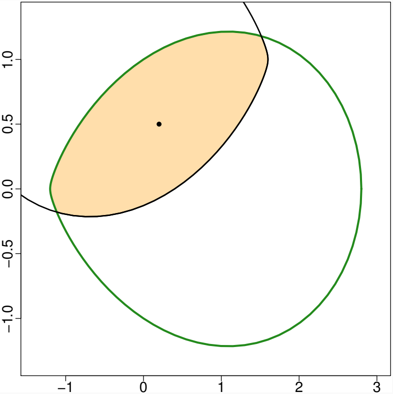

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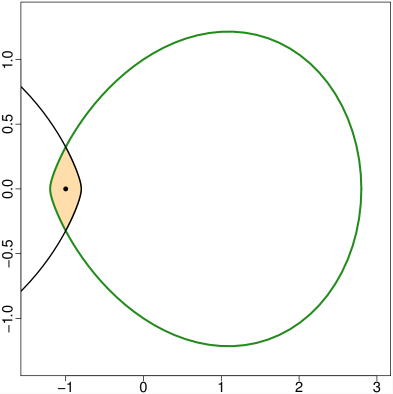

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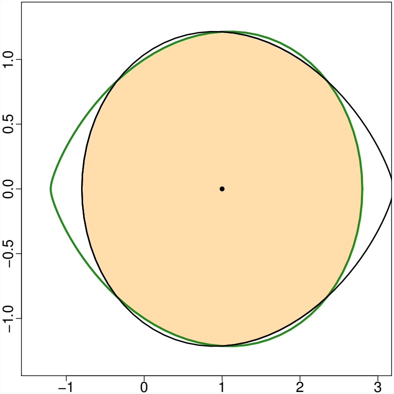

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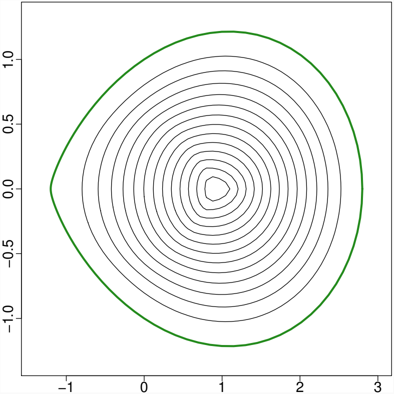

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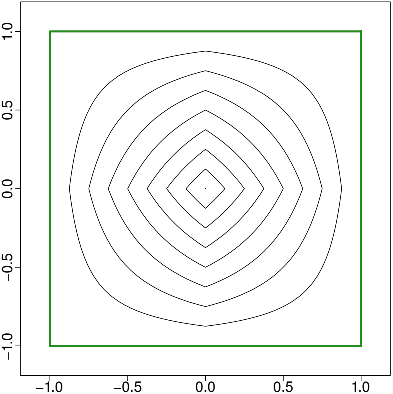

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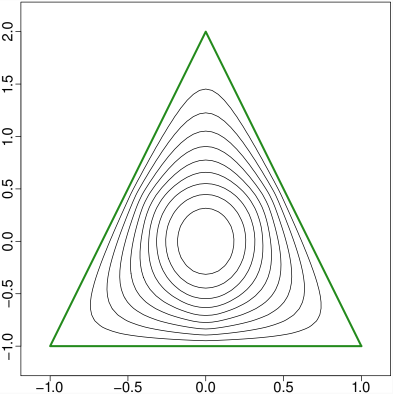
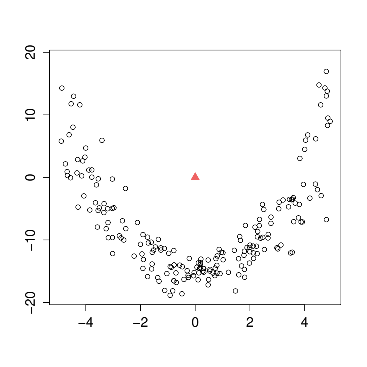
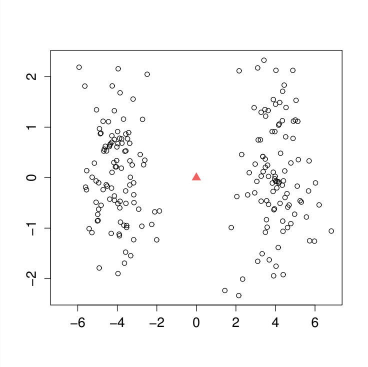


Localization of *hD* (Paindaveine and Van Bever, 2013)





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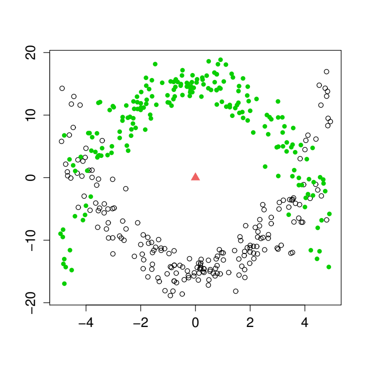

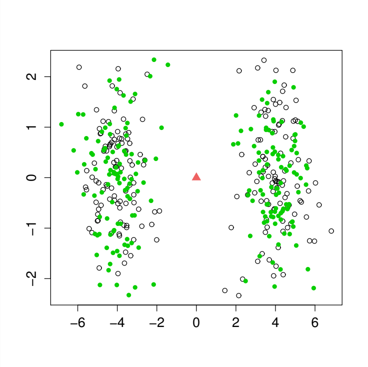

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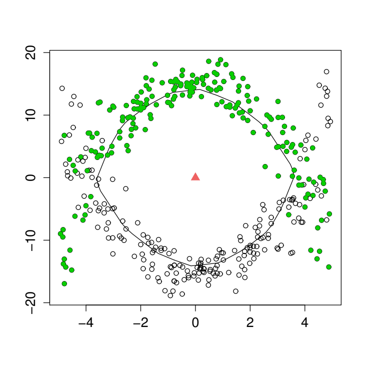

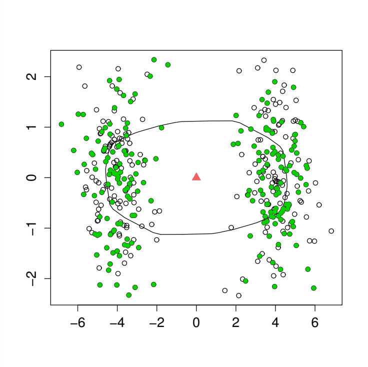

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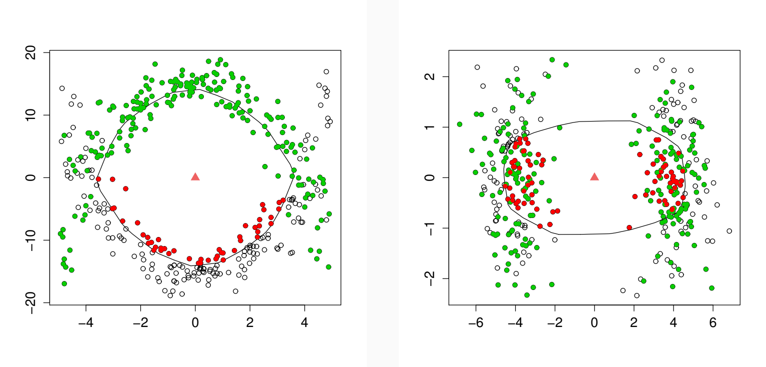

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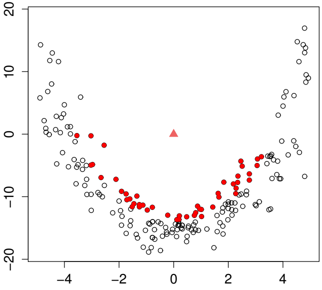

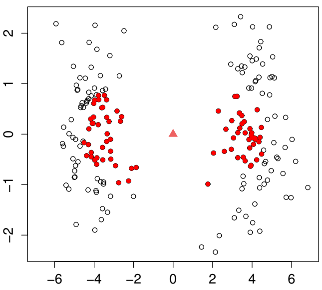

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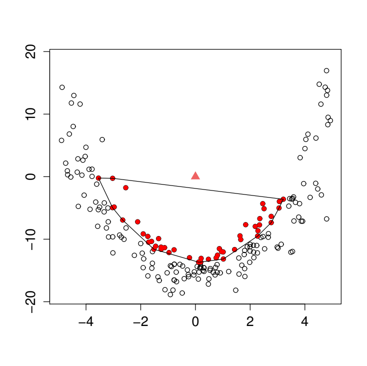

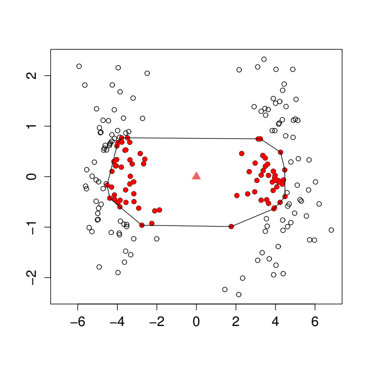

Winternitz measure of symmetry

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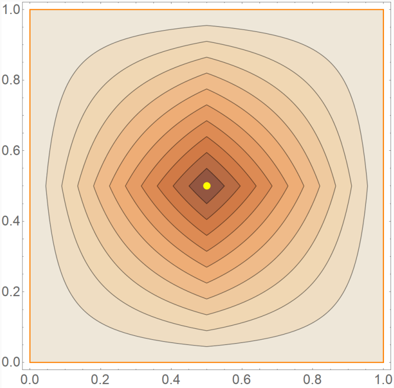

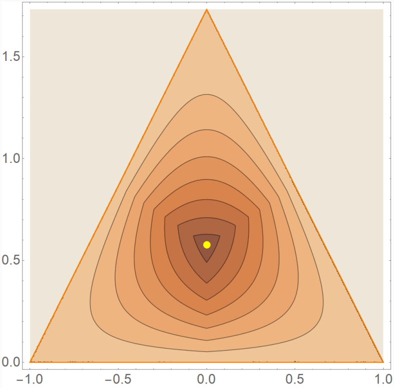

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A function $s\colon \mathcal{K}^d \to [0,1]$ is called a measure of symmetry iff

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Proposition

A convex (or star) body $K \subset \mathbb{R}^d$ is centrally symmetric iff

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From the definition of the halfspace symmetry we get

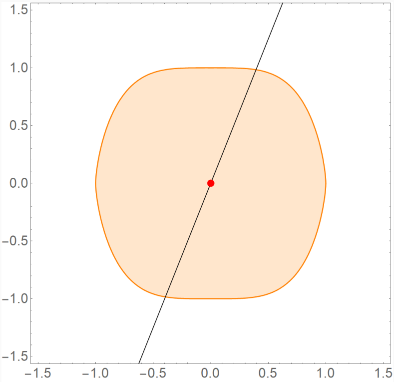
Proposition
A convex body is halfspace symmetric $\iff$ it is centrally symmetric.

Some history:

- for $d = 2$ the problem is considered trivial;
- shown for $d = 3$, and conjectured for any d by Paul Funk (1913);
- fully proved only by Schneider (1970) using functional equations;
- newer proofs involve spherical integration (Falconer, 1983);
- extensions use spherical harmonics (Groemer, 1996);
- no elementary proof known for $d > 3$.

Funk's characterization of symmetry

For $K \in \mathcal{K}^d$: H-symmetry $\iff$ A-symmetry $\iff$ C-symmetry.



Proposition (Zuo and Serfling, 2000, Theorem 2.6)
Suppose a random vector X is halfspace symmetric about a unique point $\theta \in \mathbb{R}^d$, end either

1. *X is absolutely continuous, or*
2. *X is discrete and* $\mathrm{P}(X = \theta) = 0$.

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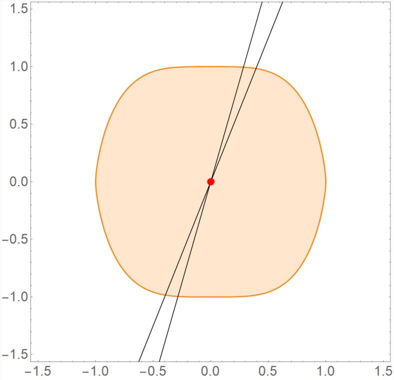
For P uniform on $K \in \mathcal{K}^d$ this implies Funk's characterization! (for convex (star) bodies angular symmetry $\equiv$ central symmetry)

Proof only for $d = 2$, for the sake of simplicity (p. 73).



Proposition (Dutta et al., 2011, Theorem 2)
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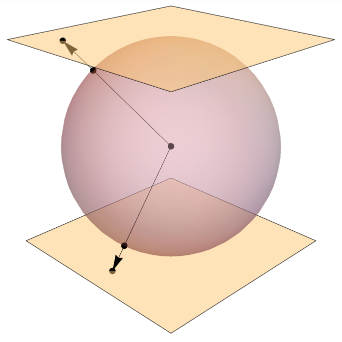
Proof II: For $d = 2$, general case "analogous".

Proposition (Rousseeuw and Ruts, 2003, Theorem 2)
If there is a point $\theta \in \mathbb{R}^d$ with

$$hD(\theta; P) = 1/2 + P(\{\theta\})/2,$$

then $X \sim P$ is angularly symmetric about θ.

(i). The map $x \mapsto (x_1/|x_d|, x_2/|x_d|, \dots, x_d/|x_d|)$ takes $\mathcal{H}(0)$ to halfspaces passing through hyperplanes $H^{\pm} = \{x \in \mathbb{R}^d : x_d = \pm 1\}$.



(ii). Apply the Cramér-Wold device (Cramér and Wold, 1936) in $\mathbb{R}^{d-1}$.

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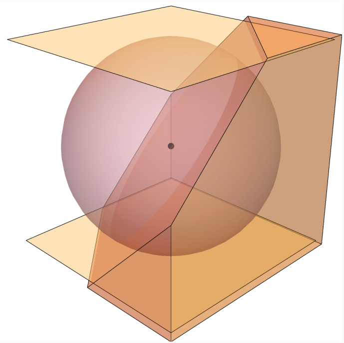

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Proof works for any d, using only the Cramér and Wold device.

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Any distribution $X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is uniquely determined by the totality of its one-dimensional projections $\langle X, u \rangle$, $u \in \mathbb{S}^{d-1}$.

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First simple proof of the characterization.

As collected by Grünbaum (1963):

- level sets of hD are convex and closed;
- for any $K \in \mathcal{K}^d$ the halfspace median is unique;

Proposition (Blaschke, 1923; Grünbaum, 1963)
For any convex body $K \subset \mathbb{R}^d$ whose halfspace median is $x \in K$, there exists a collection of at least $d+1$ halfspaces $\{H_i\}$ such that $\bigcup_i H_i = \mathbb{R}^d$, $x \in \bigcap_i H_i$, and

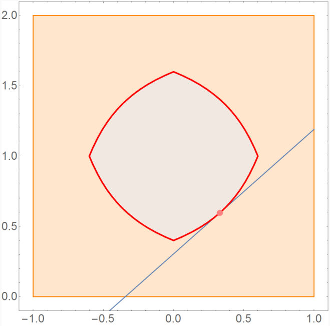
$$P(H_i) = hD(x; K) = \sup_{y \in \mathbb{R}^d} hD(y; K).$$

For each such H_i the point x is the centroid of $\partial H_i \cap K$.

Call $H \in \mathcal{H}(x)$ a minimizing halfspace of $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ at x if

$$P(H) = hD(x; P),$$

and a hyperplane ∂H a barycentric cut of P at x if the centroid of the cut (conditional expectation) of P by ∂H is x.



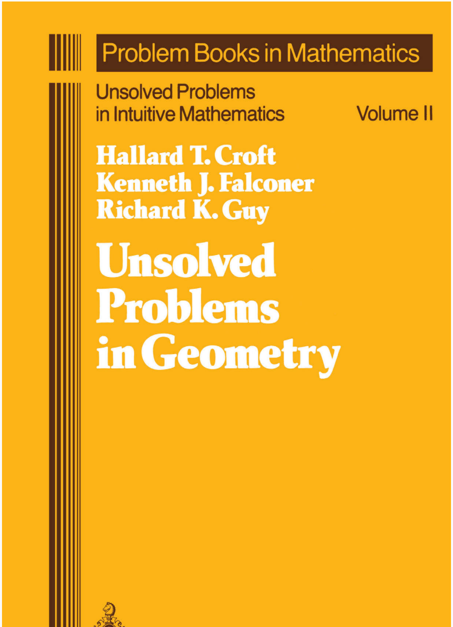
Independently, it was proved in geometry/statistics:

1. For $K \in \mathcal{K}^d$, the boundary of any minimizing halfspace is a barycentric cut (Blaschke, 1917).
2. Minimizing halfspaces of the median x of $K \in \mathcal{K}^d$ cover $\mathbb{R}^d$. (Donoho and Gasko, 1992)

Observation / Problem (Grünbaum, 1963)

For all $K \in \mathcal{K}^d$ there exist $(d+1)$ barycentric cuts through the halfspace median x of P.

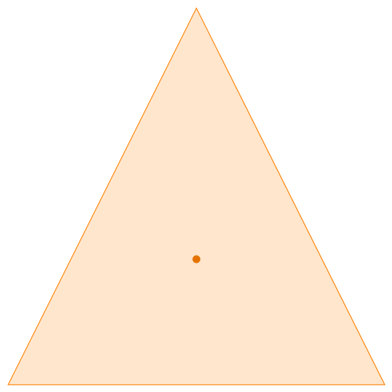
Helly's theorem for depth median



A8. Sections through the centroid of a convex body. Let K be a 3-dimensional convex body with centroid (i.e., center of gravity) $\mathbf{g}$. Is $\mathbf{g}$ necessarily the centroid of at least four plane sections of K through $\mathbf{g}$? Is it even the centroid of seven such sections, as is the case if K is a tetrahedron? More generally, if K is a d-dimensional convex body, is the centroid of K the centroid of $d + 1$ or even of $2^d - 1$ of the $(d - 1)$-dimensional sections through $\mathbf{g}$? When $d = 2$ this is easily seen to be so—in this case $\mathbf{g}$ bisects three chords of K. This question is due to Grünbaum and Loewner, see also the earlier paper by Steinhaus.

A consequence of Helly's theorem (see Section E1) is that *some* point of K is the centroid of at least $d + 1$ sections by hyperplanes. What can be said about the set of points of K enjoying this property? In the plane case Ceder showed that this set is connected, but not necessarily convex. Chakerian & Stein discuss other aspects of this problem.

Minimizing halfspaces of the median x of K cover $\mathbb{R}^d$



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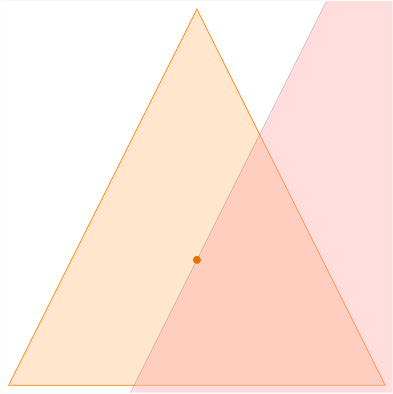

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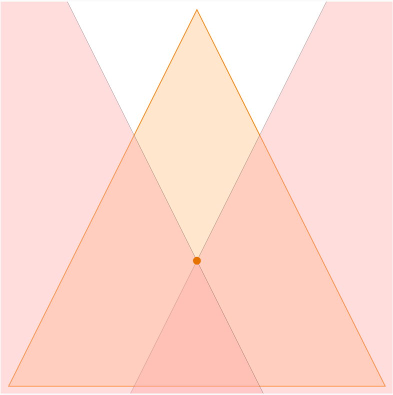

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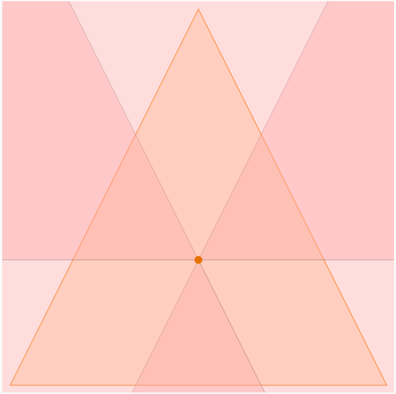
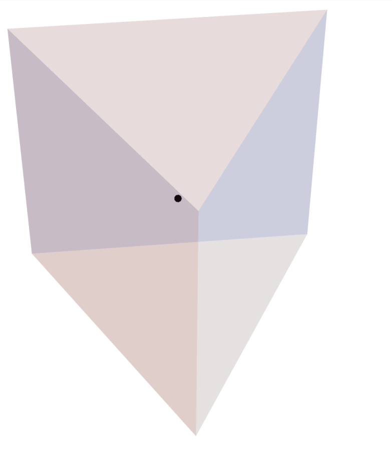


$\mathbb{R}^d$ may be covered by less than $(d+1)$ minimizing halfspaces

(Patáková, Tancer, and Wagner, 2020)



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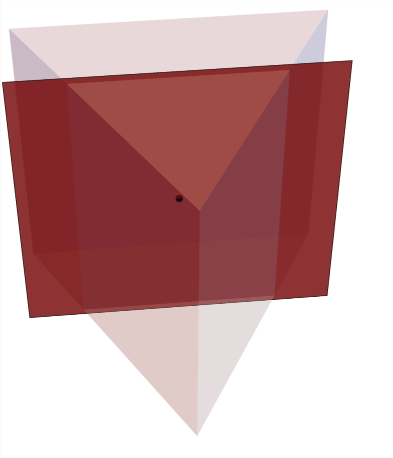

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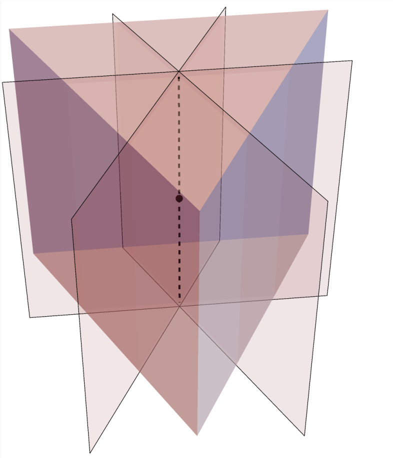

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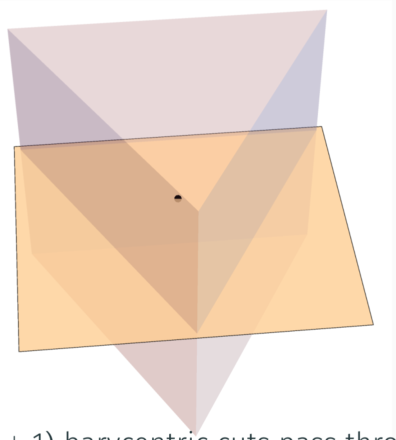

Question: Do $(d+1)$ barycentric cuts pass through the halfspace median of all $K \in \mathcal{K}^d$?

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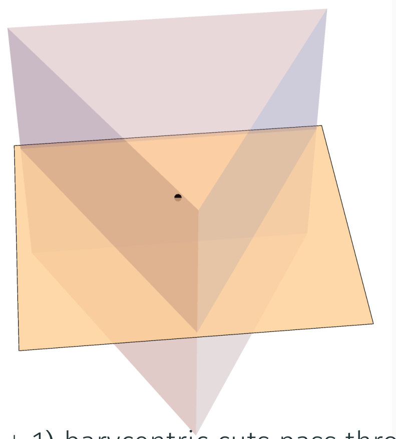
(Patáková, Tancer, and Wagner, 2020)



Question: Do $(d+1)$ barycentric cuts pass through some point for all $K \in \mathcal{K}^d$?

As collected by Grünbaum (1963):

- level sets of hD are convex and closed;
- for any $K \in \mathcal{K}^d$ the halfspace median is unique.

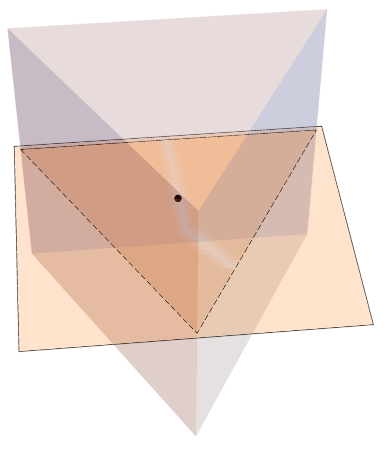
In statistics, we have

Proposition (Mizera and Volauf, 2002, Proposition 7)
Under conditions **(S)** *and* **(C)**, *the halfspace median of* $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ *is unique.*

The result is incomplete, the proof works only for $d = 2$.

Take $P \in \mathcal{P}\left(\mathbb{R}^3\right)$ the product of

uniform on a triangle in $\mathbb{R}^2$ and Cauchy in $\mathbb{R}$



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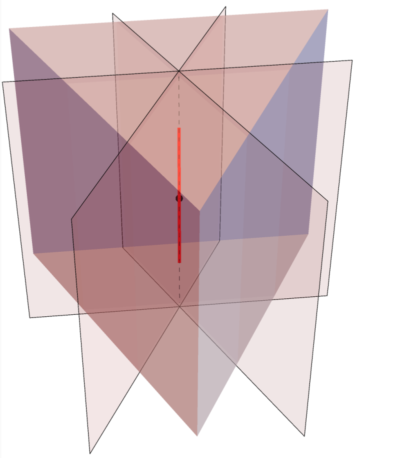


Proposition (Nagy, Pokorný, Laketa, 2021+)
A measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ has a unique median if

1. *(C) is valid,*
2. *P has a density that is "almost continuous" on hyperplanes, and*
3. *an integrability condition is satisfied (existence of expectation).*

Question: Is there a (non-convex) body $K \subset \mathbb{R}^d$ such that P uniform on K does not have a unique median?

Main messages:

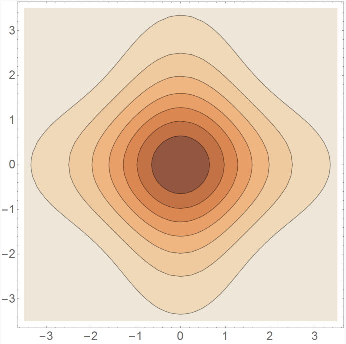
- symmetry of multivariate distributions is not an easy topic;
- Grünbaum (1960) knew about the depth before Tukey (1975);
- depth of a median is a measure of symmetry;
- many related open problems.
- Not mentioned:
  - Affine invariant points, (Grünbaum, 1963; Meyer et al., 2015, 2015b)
  - Dimensionality of depth regions. (Pokorný et al., 2021+)

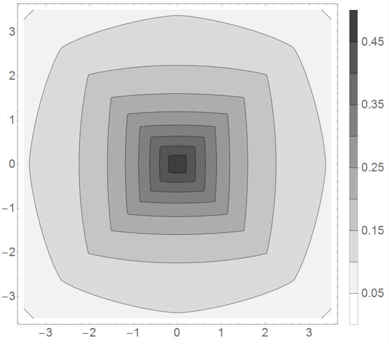
Quasi-Concavity: Floating body

Depth: Quasi-concavity

hD is always quasi-concave, i.e. for each $\delta \in [0, 1]$

$$\left\{ x \in \mathbb{R}^d : hD(x; P) \geq \delta \right\} \text{ is a convex set}$$

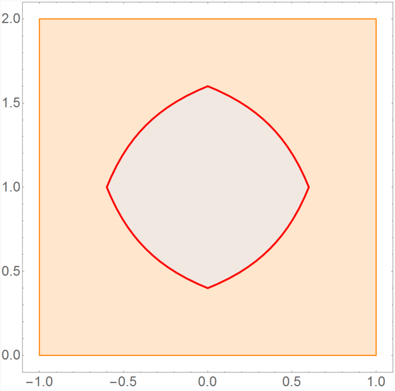




Depth: Level sets

It holds true that

$$\left\{x \in \mathbb{R}^d : hD(x; P) \geq \delta\right\} = \bigcap \{H \in \mathcal{H} : P(H) \geq 1 - \delta\}$$



Proposition (Grünbaum, 1960)
Let $K \subset \mathbb{R}^d$ be a convex body, $\mathrm{vol}(K) = 1$. *Then*

$$hD(\mathrm{E}\,K; K) \geq \left(\frac{d}{d+1}\right)^d.$$

APPLICATIONS
DE GÉOMÉTRIE
ET
DE MÉCHANIQUE;

A LA MARINE, AUX PONTS ET CHAUSSÉES, ETC.,

POUR FAIRE SUITE

AUX DÉVELOPPEMENTS DE GÉOMÉTRIE,

PAR CHARLES DUPIN,

Membre de l'Institut de France, Académie des Sciences; ancien Secrétaire de l'Académie Ionnienne, Associé étranger de l'Institut de Naples, Associé honoraire de l'Académie royale d'Irlande, et de la Société des Ingénieurs civils de la Grande-Bretagne, Membre des Académies royales des Sciences de Stockholm, de Türin, de Montpellier, etc., de la Société des Arts de Genève, de la Société d'Encouragement pour l'industrie française, Membre du Comité consultatif des Arts et Manufactures de France, Professeur de Méchanique au Conservatoire, Officier supérieur au corps du Génie Maritime, et Membre de la Légion-d'Honneur.

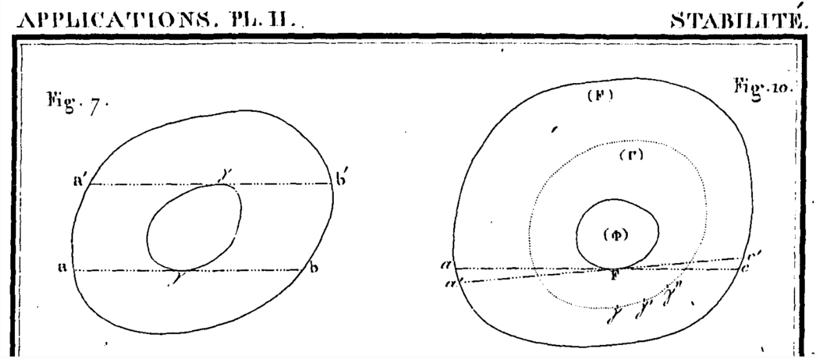
BIBLIOTHÈQUE ROYALE

PARIS,

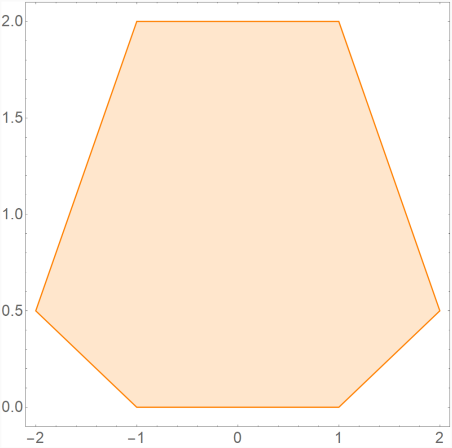
BACHELIER, SUCCESSEUR DE Mme. Ve. COURCIER, LIBRAIRE,
QUAI DES AUGUSTINS.

1822.

Definition (Dupin, 1822)
A convex body $K_{[\delta]}$ is called the floating body of $K \in \mathcal{K}^d$, if $\delta \in [0, \mathrm{vol}(K)/2]$ and each supporting hyperplane of $K_{[\delta]}$ cuts off a set of volume δ from K.



Floating body of K for $\delta = 0.2$



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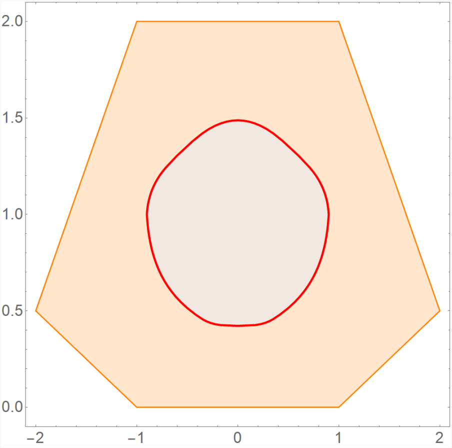

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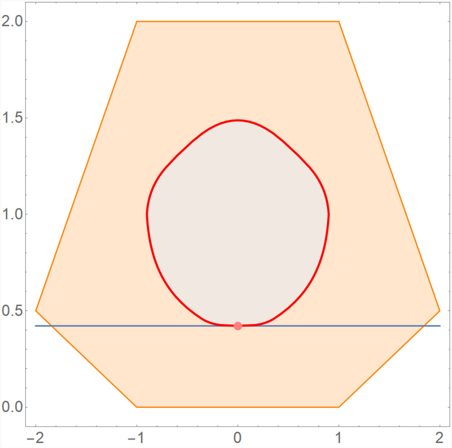

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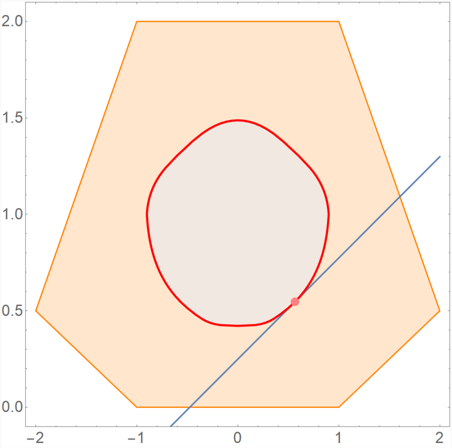
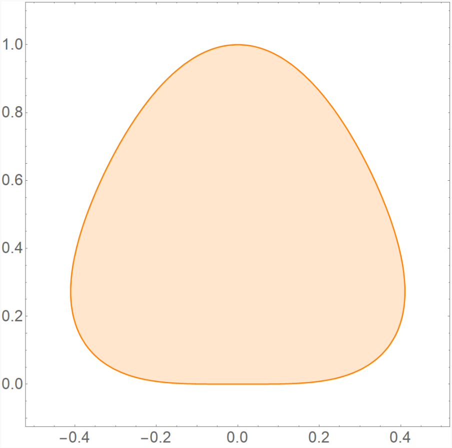


Floating body of K for $\delta = 0.3$



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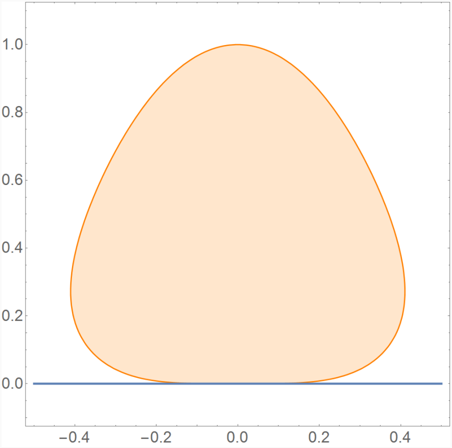

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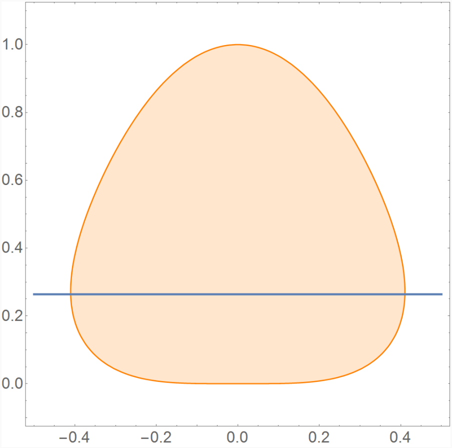

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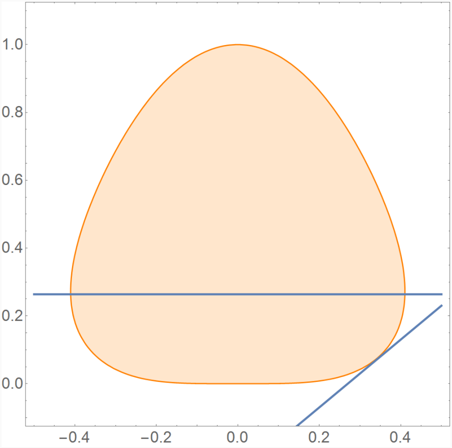

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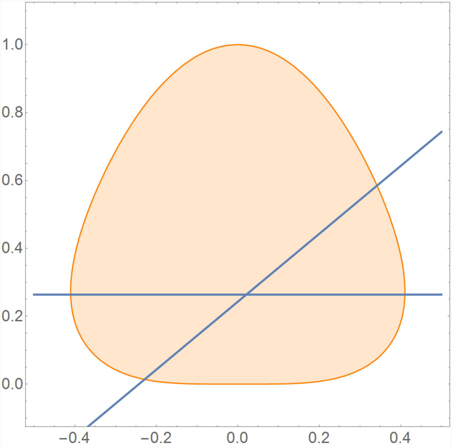

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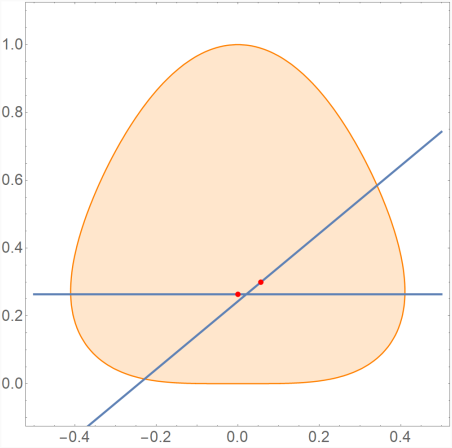

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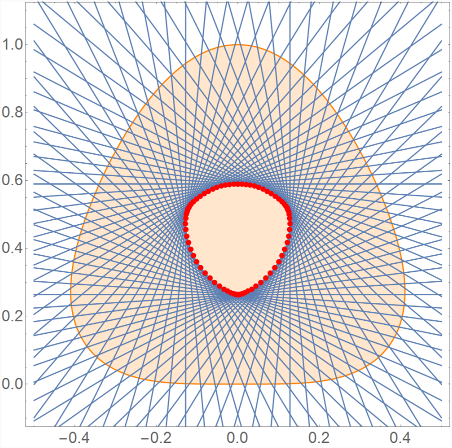
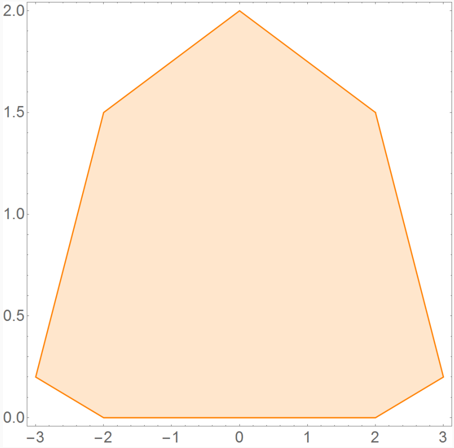
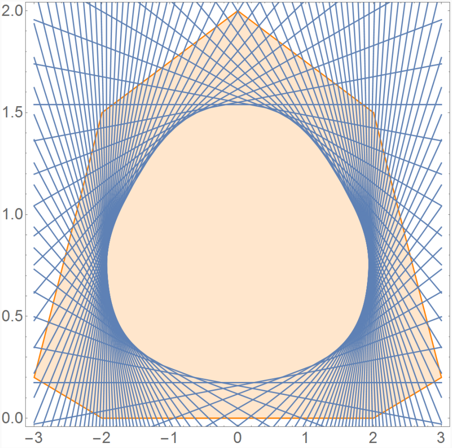


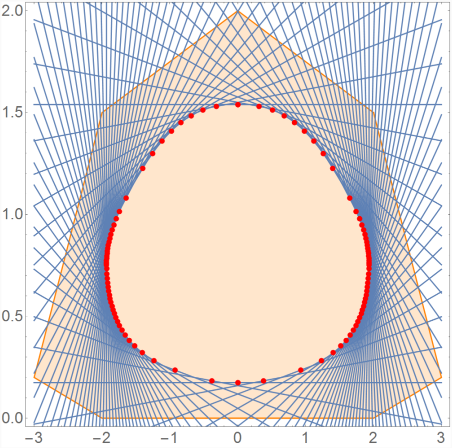
Floating body of K for $\delta = 0.1$



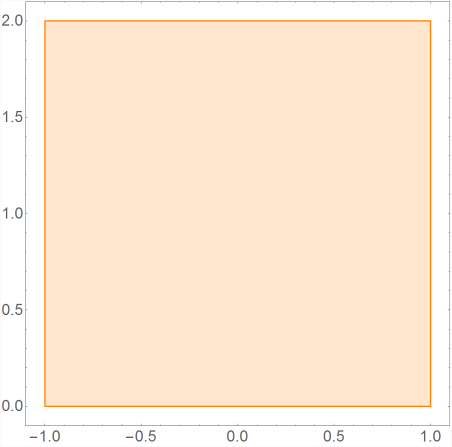
Floating body of K for $\delta = 0.1$



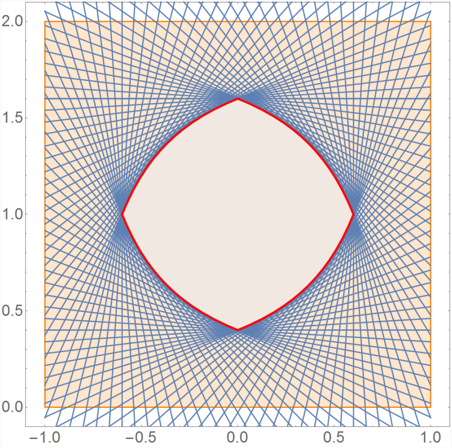
Floating body of K for $\delta = 0.1$



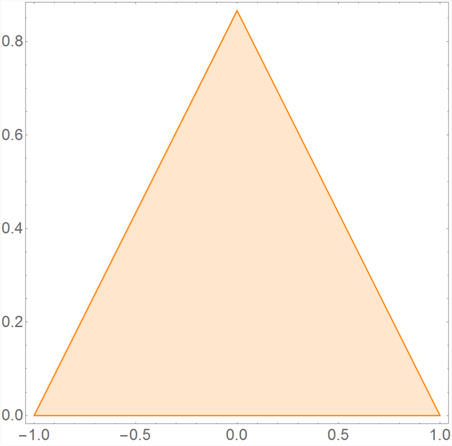
Floating body of K for $\delta = 0.3$



Floating body of K for $\delta = 0.3$



Floating body of K does not have to exist!



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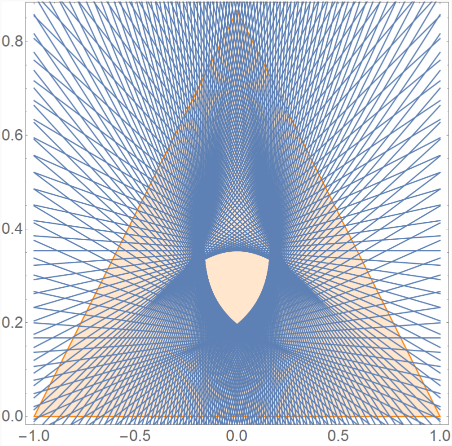

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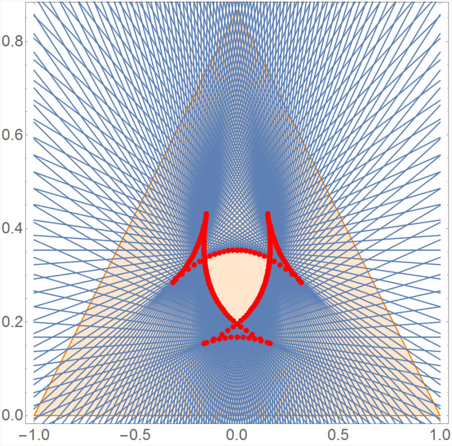

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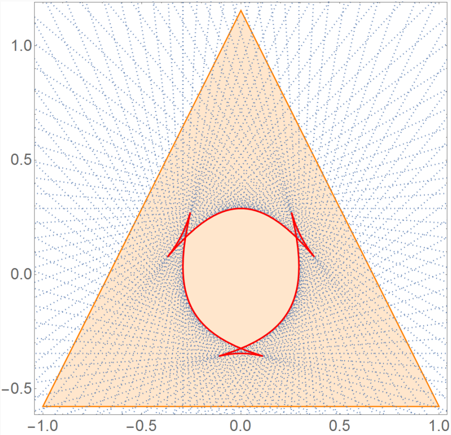

$$\Omega(K) = \int_{\partial K} \kappa(x)^{1/(d+1)} \, \mathrm{d}\,\mu(x),$$

where

- K is a convex body of class $\mathcal{C}_2^+$,
- ∂K is the topological boundary of K,
- κ is the Gauss-Kronecker curvature of K, and
- μ is the surface area measure of K
  ($d - 1$-dimensional Hausdorff measure on ∂K).

Proposition (Blaschke, 1923)
It holds true that

$$\Omega(K)^{d+1} \leq d^{d+1}\kappa_d^2 \operatorname{vol}(K)^{d-1}$$

with equality only for K ellipsoid. Here, κ_d is the volume of the unit ball in $\mathbb{R}^d$.

Ellipsoids have the largest affine surface area.

Proposition (Blaschke, 1923)
If for δ small the floating body of K exists, then for $c_d = 2\left(\kappa_{d-1}/(d+1)\right)^{2/(d+1)}$

$$\Omega(K) = \lim_{\delta \to 0} c_d \frac{\mathrm{vol}(K) - \mathrm{vol}\left(K_{[\delta]}\right)}{\delta^{2/(d+1)}}.$$

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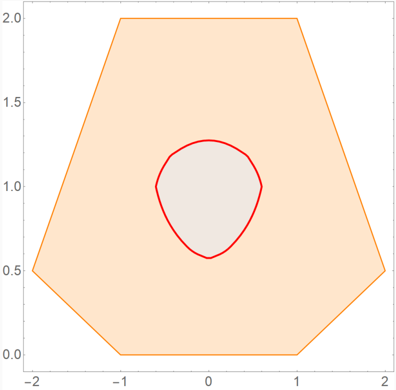

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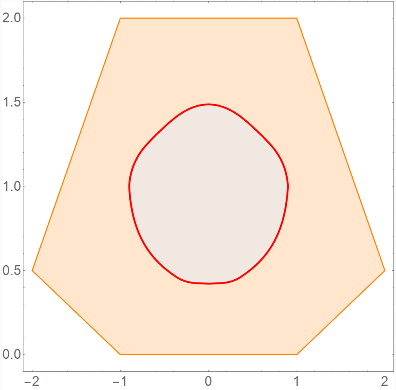

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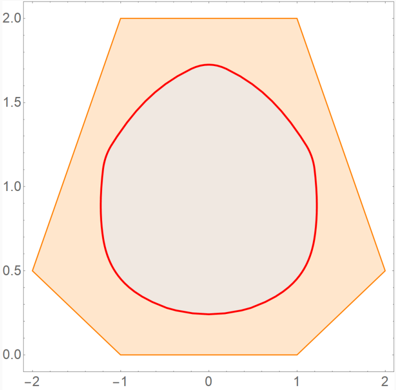
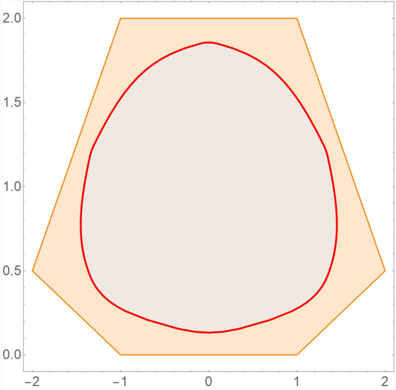


Affine surface area

$$\Omega(K) = \lim_{\delta\to 0} c_d \frac{\mathrm{vol}(K) - \mathrm{vol}\left(K_{[\delta]}\right)}{\delta^{2/(d+1)}}$$



Definition (Schütt and Werner, 1990)
Let $K \subset \mathbb{R}^d$ be a convex body and $\delta \in [0, \mathrm{vol}(K)/2]$. The convex floating body of K associated with δ is given by

$$K_\delta = \bigcap \{H \in \mathcal{H} \colon \mathrm{vol}(K \cap H) \geq 1 - \delta\}.$$

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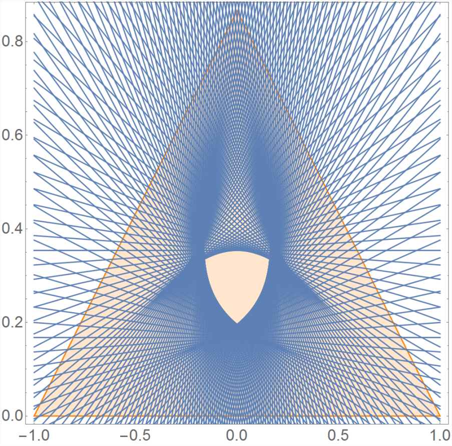
Proposition (Schütt and Werner, 1990)
K_δ always exists. If $K_{[\delta]}$ exists, then $K_{[\delta]} = K_\delta$. Further,

$$\Omega(K) = \lim_{\delta \to 0} c_d \frac{\mathrm{vol}(K) - \mathrm{vol}(K_\delta)}{\delta^{2/(d+1)}}.$$

Convex floating body of K always exists.



Definition (Bárány and Larman, 1988)
For $K \in \mathcal{K}^d$ and $x \in K$ define

$$\nu(x) = \min \left\{ \mathrm{vol}\left(K \cap H\right) : x \in H, H \in \mathcal{H} \right\}.$$

Similar functions were considered also earlier
(Neumann, 1945; Rado, 1946; Grünbaum, 1960; Leichtweiß, 1986...)

Rado (1946) defines ν in $\mathbb{R}^2$ for “densities” $f(x, y) : \mathbb{R}^2 \to [0, \infty)$.

Fresen (2012) writes about “multivariate quantiles” given by ν.

Proposition (Schütt and Werner, 1990)

$$\Omega(K) = \lim_{\delta \to 0} c_d \frac{\mathrm{vol}(K) - \mathrm{vol}(K_\delta)}{\delta^{2/(d+1)}}.$$

Problem: For measures $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ one may be interested in the behaviour of the function

$$\delta \mapsto 1 - P(\{hD(\cdot; P) \geq \delta\}) = P(\{hD(\cdot; P) < \delta\})$$

as $\delta \to 0$. How to interpret that rate of convergence?

Proposition (Fresen, 2013)
Let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ be log-concave, and let $X_1, \ldots, X_n$ be a random sample from P. Then $\operatorname{co}(X_1, \ldots, X_n)$ for $n \to \infty$ "approximates" the convex floating body of measure P corresponding to $\delta = 1/n$.

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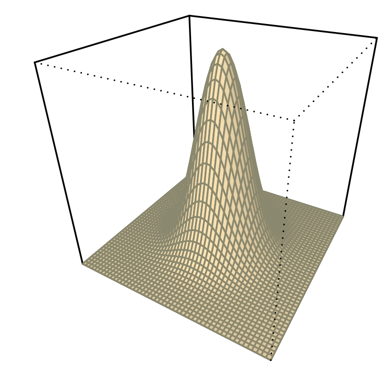
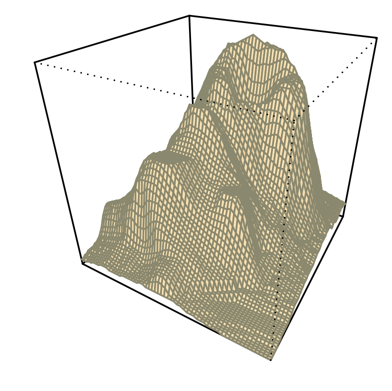
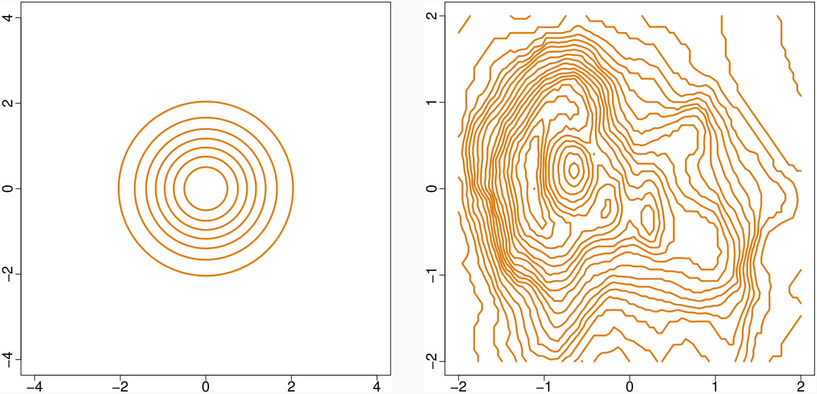
- The depth determines the rate of convergence, and the shape of the convex hull of random samples.
- Affine surface area describes the "tail complexity" of P.

$$hD_{1/n}(P) \approx \operatorname{co}(X_1, \ldots, X_n) \text{ as } n \to \infty$$

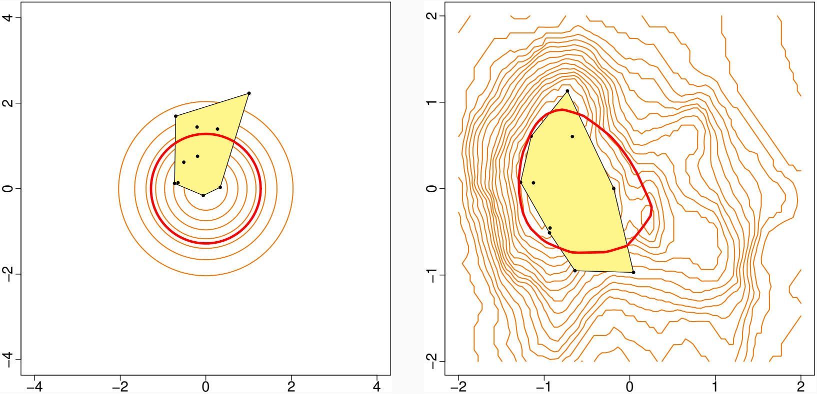




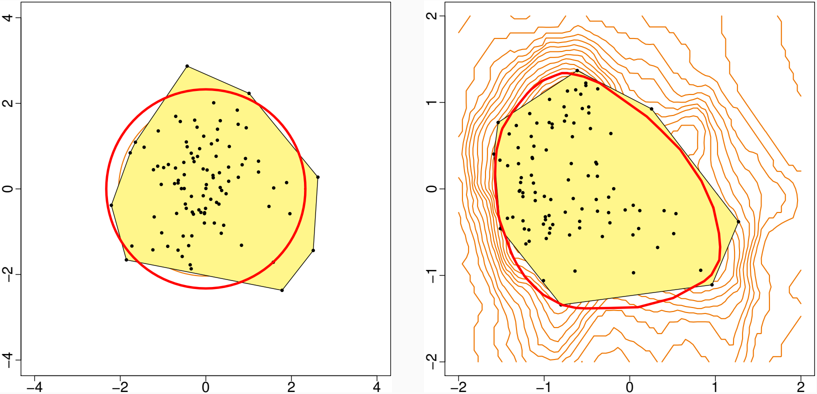
$$hD_{1/n}(P) \approx \mathrm{co}\,(X_1, \ldots, X_n) \text{ as } n \to \infty$$



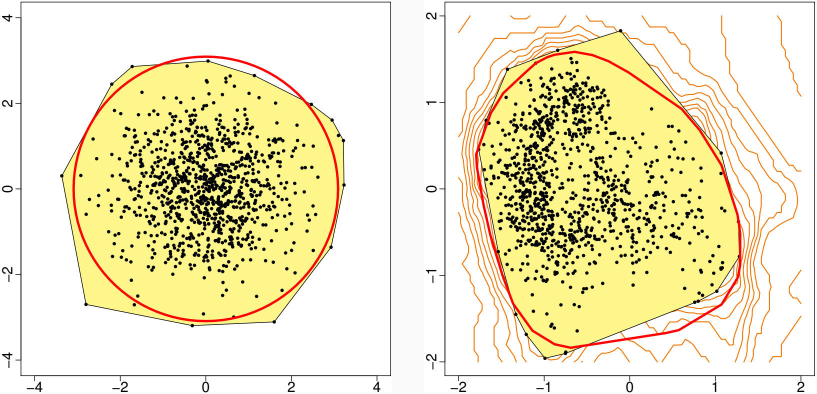
$$hD_{1/n}(P) \approx \operatorname{co}(X_1, \ldots, X_n) \text{ for } n = 10$$



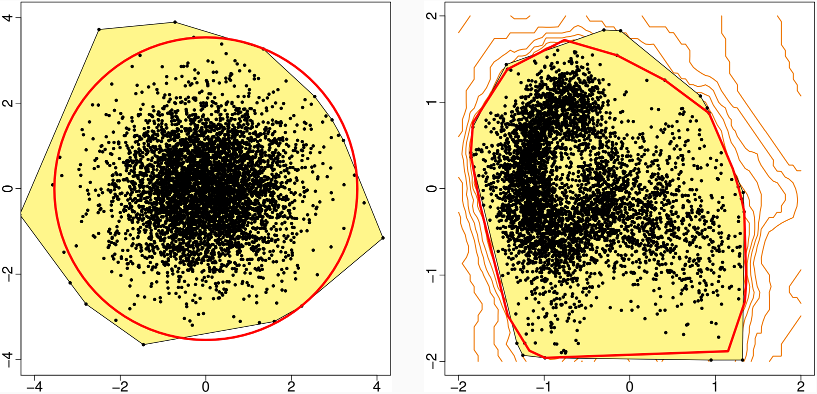
$$hD_{1/n}(P) \approx \mathrm{co}\,(X_1, \ldots, X_n) \ \text{ for } n = 100$$



$$hD_{1/n}(P) \approx \mathrm{co}\,(X_1, \ldots, X_n) \text{ for } n = 1000$$

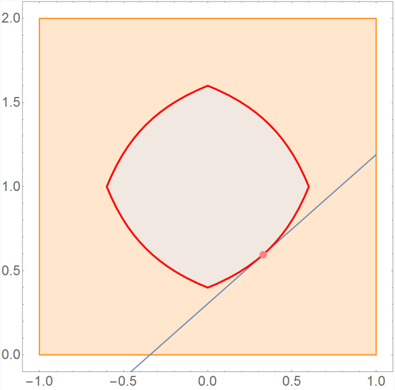


$$hD_{1/n}(P) \approx \mathrm{co}\,(X_1, \ldots, X_n) \;\text{ for } n = 5000$$



$\sqrt{n}\,(hD(x; P_n) - hD(x; P))$ is asymptotically normal

$\iff$ the contour of $hD(\cdot; P)$ is smooth in x



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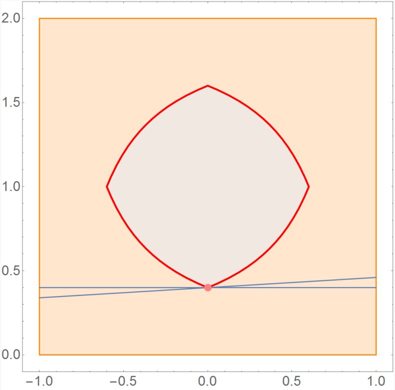
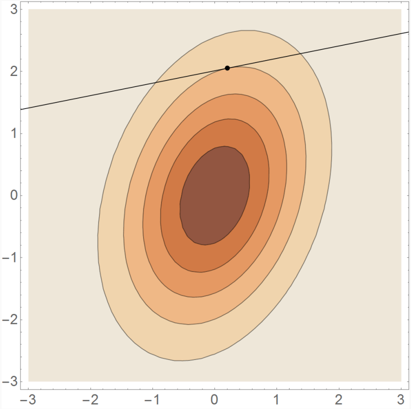


Elliptically symmetric distributions have elliptical depth contours



Problem (Massé and Theodorescu, 1994)

Does there exist a non-α-symmetric distribution with smooth depth contours?

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Does there exist a non-α-symmetric distribution with smooth depth contours?

Proposition (Meyer and Reisner, 1991)
Let K be a symmetric convex body. Then

- K_δ *is symmetric and strictly convex,*
- *if K is smooth and strictly convex, then* K_δ *is* $\mathcal{C}_2^+$.

$\Longrightarrow$ uniform distributions on smooth, symmetric, strictly convex sets have smooth depth.

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Does there exist a non-α-symmetric distribution with smooth depth contours?

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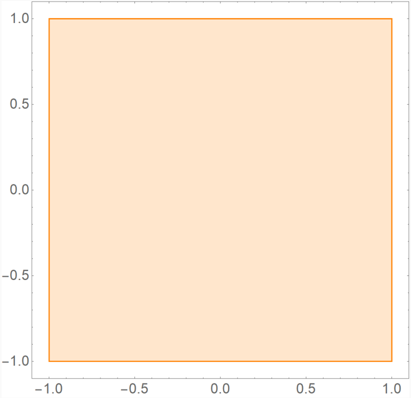
$\implies$ uniform distributions on smooth, symmetric, strictly convex sets have smooth depth.

Open problem: What can be said about general distributions with well-behaved densities?

Unit ball in L^∞ — no smooth depth contours



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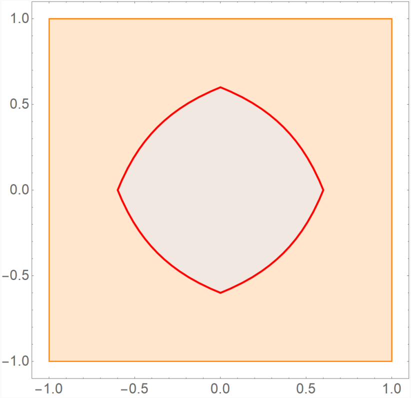
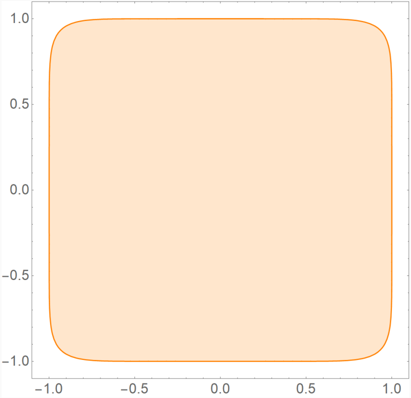


Unit ball in L^{10} — all depth contours smooth



Problem (Schütt and Werner, 1994)

Let $c, \delta > 0$ and $K = c\,K_\delta$. Is then K an ellipsoid?

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- If $K = c_n K_{\delta_n}$ for $\delta_n \to 0$ (Schütt and Werner, 1994).
- If K is $\mathcal{C}_2^+$ and $K = c\, K_\delta$ for $\delta < \delta(K)$ (Stancu, 2006, 2009).
- If $K = c\, K_\delta$ for $\delta < \delta(K)$ (Werner and Ye, 2011).

In general still an open problem.

Conjecture (Struyf and Rousseeuw, 1998)
For any $P, Q \in \mathcal{P}\left(\mathbb{R}^d\right)$, $P \neq Q$ there exists $x \in \mathbb{R}^d$ such that $hD(x; P) \neq hD(x; Q)$.

Partial positive answers: This is true if

- ~~P and Q are absolutely continuous with a compact support (Koshevoy, 2001);~~
- P and Q are empirical (Koshevoy, 2002);
- ~~P is atomic (Cuesta-Albertos and Nieto-Reyes, 2008);~~
- ~~P and Q have smooth densities (Hassairi and Regaieg, 2008);~~
- P and Q have smooth depth contours (Kong and Zuo, 2010).

Proposition (Hassairi and Regaieg, 2008, Theorem 3.2)
Let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ have a density that is smooth in the interior of its connected support. Then for any $H \in \mathcal{H}$

$$P(H) = \begin{cases} \sup_{x \in \partial H} hD(x; P) & \text{if } x_P \notin H, \\ 1 - \sup_{x \in \partial H} hD(x; P) & \text{if } x_P \in H, \end{cases}$$

where x_P is the halfspace median of P.

$\Longrightarrow$ P is characterized by its depth

Not true — can be valid only for P halfspace symmetric.

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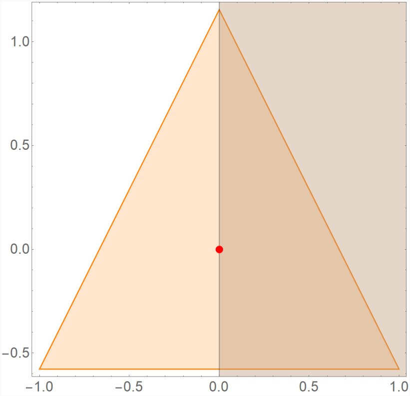

Definition
A convex body $P_{[\delta]}$ is called the floating body of a measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$, if $\delta \in [0, 1/2]$ and each supporting hyperplane of $P_{[\delta]}$ cuts off a set of probability δ.

Proposition (Nagy, Schütt, Werner, 2017)
Let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ satisfy (C), and let x_P be the halfspace median of P. Then the following are equivalent

- *For each $\delta \in (0, 1/2)$ the floating body of P exists.*
- *P satisfies (S), and for each $H \in \mathcal{H}$*

$$P(H) = \begin{cases} \sup_{x \in \partial H} hD(x; P) & \text{if } x_P \notin H, \\ 1 - \sup_{x \in \partial H} hD(x; P) & \text{if } x_P \in H. \end{cases}$$

In particular, P is characterized by its depth.

Comments:

- For any symmetric, full-dimensional, κ-concave $P \in \mathcal{P}(\mathbb{R}^d)$ with $\kappa > -1$ the floating bodies $P_{[\delta]}$ exist for all $\delta \in (0, 1/2)$; (Meyer and Reisner, 1991; Ball, 1991; Bobkov, 2010)
- Under (C):

$$
\begin{aligned}
P \text{ has smooth depth } &\implies \text{(S) and } P_{[\delta]} \text{ exist for all } \delta \\
&\implies P \text{ is H-symmetric}
\end{aligned}
$$

Conjecture (Struyf and Rousseeuw, 1998)
For any $P, Q \in \mathcal{P}\left(\mathbb{R}^d\right)$, $P \neq Q$ there exists $x \in \mathbb{R}^d$ such that $hD(x; P) \neq hD(x; Q)$.

Partial positive answers: This is true if

- P and Q are empirical
  (Struyf and Rousseeuw, 1999; Koshevoy, 2002; Laketa and Nagy, 2021);
- if all Dupin's floating bodies of P exist
  (Hassairi, Regaieg, 2008; Kong, Zuo, 2010; Nagy, Schütt, Werner, 2019).

Conjectured positive answer.
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Proposition (Nagy, 2021)
For any $d > 1$ there are two measures in $\mathcal{P}\left(\mathbb{R}^d\right)$ with the same depth.

Recall that $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is α-symmetric (Eaton, 1981) if

$$\psi(t) = \int_{\mathbb{R}^d} \exp\left(\mathrm{i}\left\langle t, x\right\rangle\right) \,\mathrm{d}\, P(x) = \xi\left(\left\|t\right\|_\alpha\right) \quad \text{for all } t \in \mathbb{R}^d$$

for some $\xi \colon \mathbb{R} \to \mathbb{R}$. For $X = (X_1, \ldots, X_d) \sim P$, these measures satisfy

$$\langle X, u\rangle \overset{d}{=} \left\|u\right\|_\alpha X_1 \quad \text{for all } u \in \mathbb{S}^{d-1}.$$

For the depth of α-symmetric P

$$hD\left(x; P\right) = \inf_{u \in \mathbb{S}^{d-1}} \mathrm{P}\left(\langle X, u\rangle \leq \langle x, u\rangle\right) = \inf_{u \in \mathbb{S}^{d-1}} \mathrm{P}\left(\left\|u\right\|_\alpha X_1 \leq \langle x, u\rangle\right)$$

$$= \mathrm{P}\left(X_1 \leq \inf_{u \in \mathbb{S}^{d-1}} \langle x, u\rangle / \left\|u\right\|_\alpha\right) = F_1\left(- \left\|x\right\|_\beta\right)$$

for β the conjugate exponent to α, and F_1 the c.d.f. of X_1.

Fix $\gamma \in (0,1)$ and take $\psi_\alpha(t) = \exp\left(- \|t\|_\alpha^\gamma\right)$ for $\gamma \leq \alpha \leq 1$. Then

- Measure P_α with characteristic function ψ_α exists (Lévy, 1937);
- The conjugate index to $\alpha \leq 1$ is $\beta = \infty$; and
- For the characteristic function of X_1 with $X \sim P_\alpha$ we have

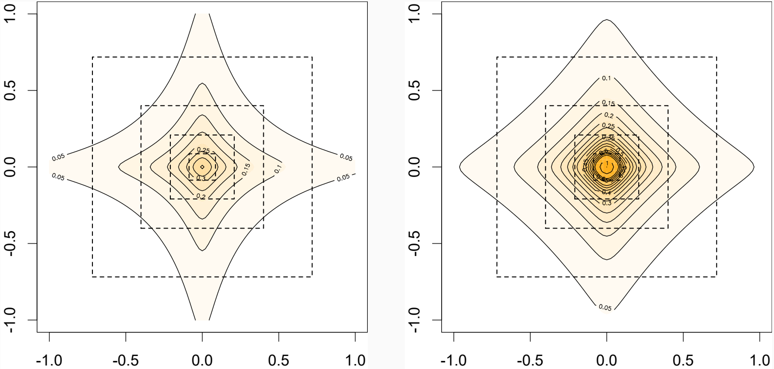
$$\mathrm{E} \exp\left(\mathrm{i}\, t X_1\right) = \exp\left(- |t|^\gamma\right) \quad \text{for all } t \in \mathbb{R},$$

  i.e. F_1 does not depend on α.

All $P_\alpha \in \mathcal{P}\left(\mathbb{R}^d\right)$ have the same depth

$$hD\left(x; P_\alpha\right) = F_1\left(- \|x\|_\infty\right) \quad \text{for all } x \in \mathbb{R}^d.$$

$\gamma = 1/2$: the density of P_α with $\alpha = 1$ (left) and $\alpha = 1/2$ (right).



Conclusions: Depth and floating bodies

What we know:

- Halfspace depth and the floating body are the same concept.
- The depth describes the asymptotics of the convex hull of samples.
- Depth does not characterize distributions.

What we do not know:

- When are floating bodies of measures Dupin's, or smooth?
- How many barycentric hyperplanes pass through medians?
- How large can the median sets be?
- When does the depth characterize distributions?

Selected Literature

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